THE ZEN OF ∞-CATEGORIES

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Abstract. In this expository essay, we provide a broad overview of abstract homotopy theory. In the interest of accessibility to a wide mathematical audience, we center our discussion around the theme of (derived) functors between abelian categories.

Contents

1. Derived categories, derived functors, and resolutions

In studying abelian categories, one immediately encounters the inescapable fact that not every functor

$$F : \mathcal{A} \to \mathcal{B}$$

among them is exact: some are only left-exact (i.e. preserve kernels), some are only right-exact (i.e. preserve cokernels), and some are neither left- nor right-exact. For example, if we take $\mathcal{A} = \mathcal{B} = \mathrm{Mod}_R$ for a commutative ring R, then for an arbitrary R-module M the functor

$$M \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R$$

will always be right-exact but will not generally be left-exact.

In his groundbreaking "Tôhoku paper" [Gro57], Grothendieck introduced an organizational framework for understanding and quantifying these failures of exactness, based on the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. This category provides a home for *resolutions* of objects of $\mathcal{A}$: these are objects which are "weakly equivalent" to our original objects of $\mathcal{A}$, but which are better behaved with respect to our given functor of interest (in a sense to be described shortly). One would now like to define the *derived functor* of F to be the value of the induced functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$$

on an appropriately chosen resolution.

However, such resolutions – and thence their values under the functor $\mathrm{Ch}(F)$ – are generally only well-defined up to weak equivalence (a/k/a "quasi-isomorphism"). There are two ways of remedying this situation.

- One may take homology of these values in $\mathrm{Ch}(\mathcal{B})$ to obtain well-defined objects of $\mathcal{B}$. For example, this technique leads to the definition of $\mathrm{Tor}_*^R(M, -)$ as the derived functor of $M \otimes_R -$.
- Alternatively, writing $\mathbf{W}_{\text{q.i.}} \subset \mathrm{Ch}(\mathcal{B})$ for the subcategory of quasi-isomorphisms, one can consider the derived functor of F as taking values in the *derived category* of $\mathcal{B}$, i.e. the localization $\mathscr{D}(\mathcal{B}) = \mathrm{Ch}(\mathcal{B})[\mathbf{W}_{\text{q.i.}}^{-1}]$.

In fact, the first approach can always be recovered from the second: by the definition of quasi-isomorphism, homology descends along the canonical localization functor $\mathrm{Ch}(\mathcal{B}) \to \mathscr{D}(\mathcal{B})$.

Of course, a derived functor should in particular be a functor, but it is not immediately obvious that the process we have described defines one. In fact, our desired functoriality will be a consequence of our definition of "resolution". The appropriate notion will vary from one application to another, but in any case the crucial property will be that the restriction

$$\mathrm{Ch}(\mathcal{A})^{\text{res}} \hookrightarrow \mathrm{Ch}(\mathcal{A}) \xrightarrow{\mathrm{Ch}(F)} \mathrm{Ch}(\mathcal{B})$$

to the full subcategory of "resolutions" preserves weak equivalences. For example, given any R-module N, any weak equivalence $P_\bullet \xrightarrow{\approx} Q_\bullet$ between projective resolutions of N induces a weak equivalence

$$M \otimes_R P_\bullet \xrightarrow{\approx} M \otimes_R Q_\bullet$$

upon tensoring with M.[1] Moreover, every object should admit a resolution: indeed, in many cases (such as with model categories, as we will see in §2), the inclusion $\mathrm{Ch}(\mathcal{A})^{\text{res}} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ even induces an equivalence

$$\mathrm{Ch}(\mathcal{A})^{\text{res}}[\mathbf{W}_{\text{q.i.}}^{-1}] \xrightarrow{\sim} \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}^{-1}] = \mathscr{D}(A)$$

[1] On the other hand, these objects are *not* generally weakly equivalent to $M \otimes_R N$: this is the entire point of resolving N in the first place.

on localizations. In such a situation, we then obtain the derived functor $\mathscr{D}(F)$ of the original functor F as an extension in the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ & & \downarrow & & \downarrow \\ \mathrm{Ch}(\mathcal{A})^{\mathrm{res}} & \hookrightarrow & \mathrm{Ch}(\mathcal{A}) & \xrightarrow{\mathrm{Ch}(F)} & \mathrm{Ch}(\mathcal{B}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}(\mathcal{A})^{\mathrm{res}}[\mathbf{W}_{\text{q.i.}}^{-1}] & \xrightarrow{\sim} & \mathscr{D}(\mathcal{A}) & \overset{\mathscr{D}(F)}{\dashrightarrow} & \mathscr{D}(\mathcal{B}) \end{array}$$

of categories (which is well-defined up to natural isomorphism). The resulting composite

$$\mathcal{A} \to \mathscr{D}(\mathcal{A}) \xrightarrow{\mathscr{D}(F)} \mathscr{D}(\mathcal{B})$$

is sometimes referred to as the *total derived functor* of F (recovering as it does the "i^{th} derived functor" of F upon postcomposition with the functor $H_i : \mathscr{D}(\mathcal{B}) \to \mathcal{B}$).

2. Model categories

By definition, the derived category $\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\text{q.i.}}^{-1}]$ of an abelian category $\mathcal{A}$ is the universal recipient of homological invariants. For example, the derived category $\mathscr{D}_R = \mathscr{D}(\mathrm{Mod}_R)$ is the target of the derived functor

$$\mathrm{Mod}_R \to \mathscr{D}_R \xrightarrow{\mathscr{D}(M \otimes_R -)} \mathscr{D}_R$$

of the functor

$$M \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Correspondingly, the derived category enjoys a universal property *as a category*. However, it tends to be quite difficult to make computations *within* the derived category. In effect, this is because its universal property takes place "one category-level higher" than do its actual objects and morphisms themselves.

In order to discuss this phenomenon, it is convenient to introduce the notion of a *relative category*: this is a (strict) category $\mathcal{R}$ equipped with a distinguished subcategory $\mathbf{W} \subset \mathcal{R}$ of "weak equivalences" which is required to contain the subcategory $\mathcal{R}^{\cong} \subset \mathcal{R}$ of isomorphisms. The category $\mathfrak{relcat}$ of relative categories admits a *localization* functor

$$\mathfrak{relcat} \to \mathfrak{cat}$$

to the category $\mathfrak{cat}$ of (strict) categories (which we have already referred to in §1), which is by definition left adjoint to the "minimal relative category structure" functor $\mathcal{C} \mapsto (\mathcal{C}, \mathcal{C}^{\cong})$. Given a relative category $(\mathcal{R}, \mathbf{W})$, its localization $\mathcal{R}[\mathbf{W}^{-1}]$ – which is also in some contexts called its "homotopy category" – is therefore equipped with a canonical localization functor

$$\mathcal{R} \to \mathcal{R}[\mathbf{W}^{-1}]$$

with the universal property that for any category $\mathcal{C} \in \mathfrak{cat}$, the restriction map

$$\hom_{\mathfrak{cat}}(\mathcal{R}[\mathbf{W}^{-1}], \mathcal{C}) \to \hom_{\mathfrak{cat}}(\mathcal{R}, \mathcal{C})$$

defines an isomorphism onto the set of functors $\mathcal{R} \to \mathcal{C}$ which take the subcategory $\mathbf{W} \subset \mathcal{R}$ of weak equivalences into the subcategory $\mathcal{C}^{\cong} \subset \mathcal{C}$ of isomorphisms.[2] At one extreme, the localization $\mathcal{R}[(\mathcal{R}^{\cong})^{-1}]$ of the minimal relative category structure is therefore simply $\mathcal{R}$ itself, while at the other extreme, the localization $\mathcal{R}[\mathcal{R}^{-1}]$ of the "maximal" relative category structure recovers the *groupoid completion* of the category $\mathcal{R}$.

Using this language, we can now illustrate the difficulty of making computations within the localization of a relative category, such as the derived category $\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\mathrm{q.i.}}^{-1}]$ of an abelian category $\mathcal{A}$.

We begin with the smallest possible example. Recall that a category with a single object is completely specified by the monoid of endomorphisms of its object; given a monoid G_0, we write $\mathfrak{B}G_0$ for the corresponding one-object category. Under this correspondence, the group completion G of the monoid G_0 corresponds to the groupoid completion of $\mathfrak{B}G_0$: that is, there is a canonical isomorphism

$$\mathfrak{B}G \cong \mathfrak{B}G_0[(\mathfrak{B}G_0)^{-1}]$$

in $\mathfrak{cat}$. But while the groupoid $\mathfrak{B}G$ is easy to characterize by means of its universal property, it is hopelessly difficult to describe in concrete terms. Indeed, understanding composition in $\mathfrak{B}G$ amounts to understanding the multiplication law of G, but this is an intractable (in fact, computationally undecidable) task, closely related to the so-called "word problem" for generators and relations in abstract algebra.

More generally, given a relative category $(\mathcal{R}, \mathbf{W})$ and any two objects $x, y \in \mathcal{R}$, morphisms from x to y in the localization $\mathcal{R}[\mathbf{W}^{-1}]$ will be represented by

[2] The term "localization functor" is certainly overloaded, but it should always be clear what is meant in any given situation.

equivalence classes of "zigzags"

$$x \overset{\approx}{\leftarrow} \bullet \rightarrow \bullet \overset{\approx}{\leftarrow} \cdots \overset{\approx}{\leftarrow} \bullet \rightarrow \bullet \overset{\approx}{\leftarrow} y$$

in $(\mathcal{R}, \mathbf{W})$ from x to y.[3,4] In particular, note that elements of $\hom_{\mathcal{R}[\mathbf{W}^{-1}]}(x,y)$ will generally fail drastically to be represented by elements of $\hom_{\mathcal{R}}(x,y)$.

It was against this backdrop that Quillen introduced the general theory of *model categories* in his seminal work [Qui67]. A model category $\mathcal{M}$ consists of a relative category equipped with certain additional data that are collectively called a *model structure*, which in particular specify full subcategories

$$\mathcal{M}^c \hookrightarrow \mathcal{M} \hookleftarrow \mathcal{M}^f$$

of *cofibrant* objects and of *fibrant* objects. Moreover, the axioms dictate that every object of $\mathcal{M}$ is weakly equivalent to a cofibrant object and is also weakly equivalent to a fibrant object. Thus, the following ***fundamental theorem of model categories*** provides a direct and computable method of accessing the hom-sets in the localization $\mathcal{M}[\mathbf{W}^{-1}]$.

Theorem 2.1 (Quillen)**.** *Let $\mathcal{M}$ be a model category, and suppose that $x \in \mathcal{M}$ is cofibrant and that $y \in \mathcal{M}$ is fibrant. Then the canonical map*

$$\hom_{\mathcal{M}}(x,y) \rightarrow \hom_{\mathcal{M}[\mathbf{W}^{-1}]}(x,y)$$

is a surjection, which moreover becomes an isomorphism after applying either equivalence relation of "left homotopy" or "right homotopy" to the source.

Thus, cofibrant objects should be thought of as being "good for mapping out of", while fibrant objects should be thought of as being "good for mapping into".[5]

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[3]Strictly speaking, we should really be referring to the *images* of x and y under the localization functor $\mathcal{R} \rightarrow \mathcal{R}[\mathbf{W}^{-1}]$, but (since we are speaking strictly) this induces an isomorphism on sets of objects and so there is no real ambiguity.

[4]As an example of the equivalence relation on zigzags, if one of the backwards-pointing weak equivalences happens to be an isomorphism, then the displayed zigzag must be declared equivalent to the one obtained by replacing this weak equivalence with its (forwards-pointing) inverse and then composing with any adjacent forwards-pointing arrows.

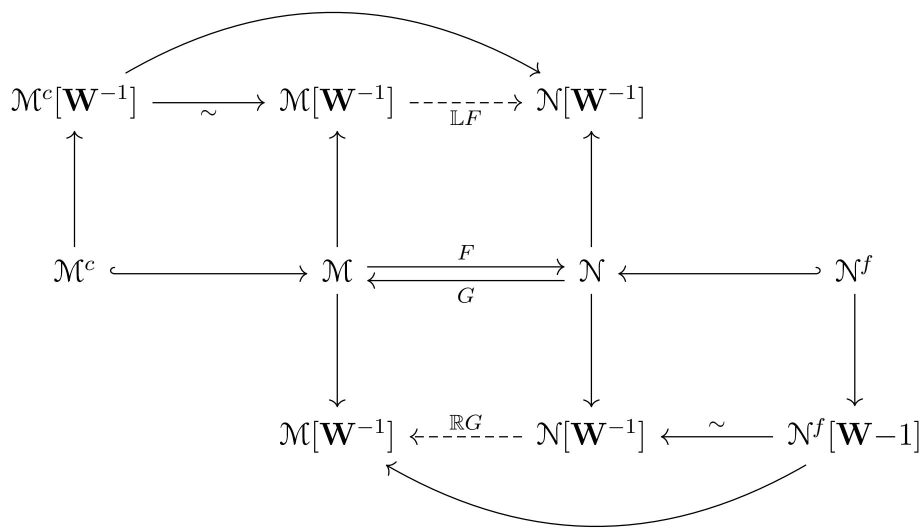
[5]For example, the relative category $(\mathrm{Ch}_R, \mathbf{W}_{\text{q.i.}})$ admits a model structure in which bounded-below complexes of projective R-modules are cofibrant and all objects are fibrant, and the "homotopy" relations can be computed via the usual notion of chain homotopy. In fact, this same relative category admits *another* model structure, in which bounded-above complexes of injectives are fibrant and all objects are cofibrant. The existence of these two distinct model structures is responsible e.g. for the fact that we can compute $\mathrm{Ext}^*_R(M,N)$ either by applying the functor $\hom_R(-,N)$ to a projective resolution $P_\bullet \overset{\approx}{\rightarrow} M$ or by applying the functor $\hom_R(M,-)$ to an injective resolution $N \overset{\approx}{\rightarrow} I^\bullet$.

Beyond providing direct access to computations in derived categories, the theory of model categories moreover bears directly on the construction of derived functors. Given two model categories $\mathcal{M}$ and $\mathcal{N}$, a *Quillen adjunction* between them is an adjunction

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

satisfying certain conditions related to their respective model structures, and a *Quillen equivalence* is a Quillen adjunction satisfying a further condition. These notions are immensely useful for both constructing and computing derived functors, as a consequence of the following result.

Theorem 2.2 (Quillen)**.** *Given a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$, the left adjoint F preserves weak equivalences between cofibrant objects of $\mathcal{M}$ and the right adjoint G preserves weak equivalences between fibrant objects of $\mathcal{N}$. These induce derived functors via the commutative diagram*



of categories, which moreover participate in a canonical ***derived adjunction***

$$\mathbb{L}F : \mathcal{M}[\mathbf{W}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}^{-1}] : \mathbb{R}G$$

on localizations. If this Quillen adjunction is moreover a Quillen equivalence, then the derived adjunction is an adjoint equivalence of categories.

Thus, more generally, cofibrant objects should be thought of as *left resolutions*, while fibrant objects should be thought of as *right resolutions*.

3. Nonabelian derived categories

Let us return to our discussion of functors $F : \mathcal{A} \to \mathcal{B}$ between abelian categories. Recall that in good cases, the derived functor of F can be computed by passing to the induced functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$$

on categories of chain complexes and then restricting to a subcategory of "resolutions" (whose precise nature depends on the situation at hand).

Let us restrict our attention for a moment to the subcategory $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \subset \mathrm{Ch}(\mathcal{A})$ of nonnegatively-graded chain complexes. Then, there is an equivalence

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \simeq s\mathcal{A}$$

with the category of *simplicial objects* in $\mathcal{A}$, i.e. the category $s\mathcal{A} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ of $\mathcal{A}$-valued presheaves on the category $\mathbf{\Delta}$ of finite nonempty totally-ordered sets.

This leads to an enormously fruitful idea: if we are interested in resolving objects of a *nonabelian* category $\mathcal{C}$, then the category $s\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ provides a reasonable substitute for the (nonexistent) category of "nonnegatively-graded chain complexes in $\mathcal{C}$". Moreover, the category $s\mathcal{C}$ still comes equipped with a subcategory $\mathbf{W} \subset s\mathcal{C}$ of weak equivalences (which reduces to that of quasi-isomorphisms in the abelian case), allowing us to form the *nonnegatively-graded nonabelian derived category* of $\mathcal{C}$ as the localization

$$\mathscr{D}_{\geq 0}(\mathcal{C}) = s\mathcal{C}[\mathbf{W}^{-1}].$$ [6,7]

As a first example, let us take $\mathcal{C} = \mathcal{S}\mathrm{et}$ to be the category of sets. Now, the category $s\mathcal{S}\mathrm{et}$ of simplicial sets admits a *geometric realization* functor

$$|-| : s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$$

to the category of topological spaces: this uses a simplicial set as a recipe for assembling a simplicial complex, with the structure maps between the various constituent sets specifying the gluing data between topological simplices. In this case, the subcategory $\mathbf{W} \subset s\mathcal{S}\mathrm{et}$ is pulled back from the subcategory $\mathbf{W}_{\mathrm{w.h.e.}} \subset \mathcal{T}\mathrm{op}$ of weak homotopy equivalences, and moreover the geometric realization functor induces an equivalence

$$\mathscr{D}_{\geq 0}(\mathcal{S}\mathrm{et}) = s\mathcal{S}\mathrm{et}[\mathbf{W}^{-1}] \xrightarrow{\sim} \mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}].$$

In other words, the nonnegatively-graded nonabelian derived category of sets is nothing other than the classical homotopy category of topological spaces!

[6] Actually, this is all a very slight simplification of the situation: one must choose a subcategory $\mathcal{G} \subset \mathcal{C}$ of "projective generators", and this determines the subcategory $\mathbf{W} \subset s\mathcal{C}$. However, we will elide this point.

[7] In fact, it is also possible to define the *full* nonabelian derived category $\mathscr{D}(\mathcal{C})$ through a more elaborate construction – the key notion is that of *spectrum objects* (in the sense of stable homotopy theory) – but we will not pursue that here.

In this sense, the category $s\mathcal{S}\mathrm{et}$ of simplicial sets can be seen as a combinatorial presentation of the homotopy category $\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$ of topological spaces, and simplicial sets themselves can be seen as combinatorial presentations of homotopy types.

In fact, the geometric realization functor participates in an adjunction

$$|-| : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}(-)_\bullet$$

with the *singular simplicial set* functor: given a topological space $X \in \mathcal{T}\mathrm{op}$, the corresponding simplicial set $\mathrm{Sing}(X)_\bullet \in s\mathcal{S}\mathrm{et} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et})$ is given by taking the object $[n] = \{0, \ldots, n\} \in \mathbf{\Delta}^{op}$ to the set

$$\mathrm{Sing}_n(X) = \hom_{\mathcal{T}\mathrm{op}}(\Delta^n_{\mathrm{top}}, X)$$

of continuous maps into X from the standard topological n-simplex Δ^n_{top}.[8] Moreover, there exist model structures on these two relative categories – the *Kan–Quillen model structure* on $s\mathcal{S}\mathrm{et}$ and the *Quillen–Serre model structure* on $\mathcal{T}\mathrm{op}$ – making this adjunction into a Quillen equivalence.[9] In particular, the category $s\mathcal{S}\mathrm{et}$ is not merely a combinatorial presentation of the homotopy category $\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$: its Kan–Quillen model structure moreover allows for extremely efficient computations therein.

4. The homotopy theory of homotopy theories

In their radical and innovative paper [DK80], Dwyer–Kan turned the lens of abstract homotopy theory onto itself, introducing a *derived functor* of the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$: this is a functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$$

[8]The functoriality of $\mathrm{Sing}(X)_\bullet : \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ arises from pulling back along certain continuous functions between the various topological simplices, which are defined by mimicking the behavior of the corresponding morphisms in $\mathbf{\Delta}$ on vertices and then extending linearly. In fact, these assemble into a *cosimplicial* object $\Delta^\bullet_{\mathrm{top}} : \mathbf{\Delta} \to \mathcal{T}\mathrm{op}$, and we can consider the functor

$$\mathrm{Sing}(-)_\bullet = \hom^{\mathrm{lw}}_{\mathcal{T}\mathrm{op}}(\Delta^\bullet_{\mathrm{top}}, -)$$

as arising from taking "levelwise maps" out of this cosimplicial topological space.

[9]The derived left adjoint of this Quillen equivalence recovers the equivalence described above: all objects of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are cofibrant, so the restriction to the subcategory $s\mathcal{S}\mathrm{et}^c_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ (as in the statement of Theorem 2.2) is already implicit.

landing in the category of *simplicially-enriched categories*, i.e. in the category of categories enriched in the category $s\mathcal{S}\mathrm{et}$ of simplicial sets.[10,11] As simplicial sets can be considered as presentations of homotopy types, objects of $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ can be considered as presentations of "categories enriched in homotopy types".[12]

Of course, such a viewpoint immediately suggests a notion of "weak equivalence" among simplicially-enriched categories; weak equivalence classes of objects of $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ came to be known colloquially as *homotopy theories*, and the corresponding localization

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$$

came to be known as *the homotopy theory of homotopy theories*.

As we have seen, such a definition is of rather limited use in and of itself: it is generally extremely difficult to make computations in a localization. However, in [Ber07], Bergner drastically improved the state of affairs by constructing a model structure on $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ extending this relative category structure, providing the first *model category* presenting the homotopy theory of homotopy theories.

Given a relative category $(\mathcal{R}, \mathbf{W})$, we denote its derived localization – also known as its *underlying homotopy theory* – by

$$\mathcal{R}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}].$$

This power series notation is meant to indicate that the derived localization contains "higher-order" information than does the ordinary localization $\mathcal{R}[\mathbf{W}^{-1}] \in \mathcal{C}\mathrm{at}$. Indeed, the "homotopy category" functor

$$\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to \mathcal{C}\mathrm{at}$$

(which takes each hom-simplicial set (considered as a homotopy type) to its set of path components) takes the subcategory $\mathbf{W} \subset \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ of weak equivalences into the subcategory $\mathbf{W} \subset \mathcal{C}\mathrm{at}$ of *equivalences* of categories, and the induced

[10] Simplicially-enriched categories are not quite the same thing as simplicial objects in $\mathcal{C}\mathrm{at}$: rather, $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \subset s(\mathcal{C}\mathrm{at})$ defines a full subcategory on those objects whose "simplicial set of objects" is constant.

[11] Their technique falls squarely in line with the "simplicial objects as resolutions" paradigm described in §3: the derived localization functor is defined as the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to s(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}) \to \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$$

of a "free simplicial resolution" functor followed by a levelwise application of the ordinary localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$.

[12] Actually, this is not quite correct: a simplicially-enriched category also contains "homotopy-coherence data" for its composition (in a sense to be described in §5) which are not present in a category enriched in homotopy types.

diagram

$$\begin{array}{ccc} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} & \xrightarrow{(\mathcal{R},\mathbf{W})\mapsto\mathcal{R}[\![\mathbf{W}^{-1}]\!]} & \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}] \\ & \searrow^{(\mathcal{R},\mathbf{W})\mapsto\mathcal{R}[\mathbf{W}^{-1}]} & \downarrow \mathrm{ho} \\ & & \mathcal{C}\mathrm{at}[\mathbf{W}^{-1}] \end{array}$$

commutes (up to natural isomorphism).

5. The zen of ∞-categories

Since the work of Bergner, there has been a proliferation of model categories which are Quillen equivalent to $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ and thus likewise present the homotopy theory of homotopy theories (by virtue of Theorem 2.2). Purely as a matter of terminology, objects of *any* of these model categories – or more precisely, their weak equivalence classes – have come to be referred to as *∞-categories*.

In fact, some of these other model categories of ∞-categories enjoy better technical properties than does $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ (or does its close cousin $(\mathcal{C}\mathrm{at}_{\mathcal{T}\mathrm{op}})_{\mathrm{Bergner}}$), making them far more useful in practice.[13] However, in addition to these technical advantages, certain of these other model categories admit *philosophical* advantages. In essence, the idea is that ∞-categories should not really be thought of as being *strictly* enriched – in topological spaces, or simplicial sets, or anything else: rather, they should be thought of as being enriched *in the ∞-category of spaces*, namely the equivalence class

$$\mathcal{S} \in \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$$

of either equivalent derived localization

$$\mathcal{T}\mathrm{op}[\![\mathbf{W}^{-1}_{\mathrm{w.h.e.}}]\!] \simeq s\mathcal{S}\mathrm{et}[\![\mathbf{W}^{-1}_{\mathrm{KQ}}]\!]$$

[13] In essence, the issue is that the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ behaves poorly with respect to products: the product of two cofibrant objects will not generally be cofibrant. This is a major issue, for it obstructs a clean construction of a "homotopically correct" internal hom-object. At the level of the homotopy category $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$, this should be an object $\underline{\hom}(\mathcal{C},\mathcal{D})$ with represented functor given by

$$\hom_{\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]}(\mathcal{E},\underline{\hom}(\mathcal{C},\mathcal{D})) \cong \hom_{\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]}(\mathcal{E}\times\mathcal{C},\mathcal{D}).$$

But since it's not straightforward to obtain a cofibrant representative of the product $\mathcal{E}\times\mathcal{C}$ at the level of the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$, it becomes difficult to naturally construct an object at that level that descends through the localization $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]$ to represent the functor $\hom_{\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}[\mathbf{W}^{-1}]}((-)\times\mathcal{C},\mathcal{D})$.

of a relative category.[14] In other words, the ∞-category $\mathcal{S}$ of spaces plays an analogous role in ∞-category theory to the one played by the category $\mathcal{S}\mathrm{et}$ in 1-category theory. In order to illustrate this idea, we briefly survey the theory of *quasicategories*.

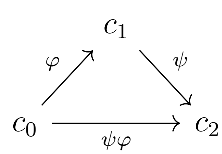
We begin by recalling the *nerve* construction, which is a functor

$$\mathrm{N}(-)_\bullet : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}.$$

By definition, the category $\mathbf{\Delta}$ is a category of posets, which are particular examples of categories; thus there is an inclusion functor $\mathbf{\Delta} \hookrightarrow \mathcal{C}\mathrm{at}$. Then, the nerve functor is given by the restricted Yoneda embedding: for any $\mathcal{C} \in \mathcal{C}\mathrm{at}$ and any $[n] \in \mathbf{\Delta}$, we define

$$\mathrm{N}(\mathcal{C})_n = \hom_{\mathcal{C}\mathrm{at}}([n], \mathcal{C}).$$

So the set $\mathrm{N}(\mathcal{C})_n$ of n-simplices is the set of sequences of n composable morphisms in $\mathcal{C}$ (with $\mathrm{N}(\mathcal{C})_0$ simply the set of objects), and for instance the morphism $\{0,1\} \to \{0,1,2\}$ in $\mathbf{\Delta}$ given by $0 \mapsto 0$ and $1 \mapsto 2$ determines a function $\mathrm{N}(\mathcal{C})_2 \to \mathrm{N}(\mathcal{C})_1$ which takes a pair of composable morphisms $(c_0 \xrightarrow{\varphi} c_1, c_1 \xrightarrow{\psi} c_2)$ to its composite $(c_0 \xrightarrow{\psi\varphi} c_2)$. Thus, a 2-simplex of $\mathrm{N}(\mathcal{C})_\bullet$ may be thought of as encoding a commutative triangle



in $\mathcal{C}$, and we may therefore think of it as a "witness" to the fact that the morphism $\psi\varphi$ is the composite of the morphisms φ and ψ. As composition in the category $\mathcal{C}$ is uniquely defined, it follows that for any two "composable 1-simplices" of $\mathrm{N}(\mathcal{C})_\bullet$ (such as φ and ψ as above), there exists a *unique* 2-simplex extending them (such as the 2-simplex above).

Now, in the setting of simplicially-enriched categories, the nerve functor can be enhanced to the *homotopy-coherent nerve* functor, denoted

$$\mathrm{N}^{\mathrm{hc}}(-)_\bullet : \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to s\mathcal{S}\mathrm{et}.$$

Rather than describe this in full, we will simply indicate its values in the bottom few dimensions. For a simplicially-enriched category $\mathcal{C} \in \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$, we once again have that the set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_0$ of 0-simplices is given by the set of objects of $\mathcal{C}$, and that the set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_1$ of 1-simplices is given by the set of morphisms of $\mathcal{C}$ (i.e. the 0-simplices of its various hom-simplicial sets, or equivalently the morphisms in its underlying unenriched category). However, the set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_2$

[14] In fact, the ∞-category of spaces admits various universal characterizations which make no reference whatsoever to topological spaces or to simplicial sets: for instance, it is the *free cocompletion* of the terminal ∞-category.

of 2-simplices is more interesting: for any three morphisms $c_0 \xrightarrow{\varphi} c_1$, $c_1 \xrightarrow{\psi} c_2$, and $c_0 \xrightarrow{\rho} c_2$ in $\mathcal{C}$, a 2-simplex

$$\begin{array}{ccccc} & & c_1 & & \\ & {\scriptstyle\varphi}\nearrow & & \searrow{\scriptstyle\psi} & \\ c_0 & & \xrightarrow[\rho]{} & & c_2 \end{array}$$

is determined by a *1-simplex* in the simplicial set $\underline{\hom}_{\mathcal{C}}(c_0, c_2)$ connecting the 0-simplices $\psi\varphi$ and ρ. Thinking of such a 1-simplex as a "path" in this "hom-space", we may therefore think of such a 2-simplex as a witness to the *homotopy commutativity* of this triangle.

Of course, one such 2-simplex of $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$ can be obtained simply by taking $\rho = \psi\varphi$ (and by taking the "path" to be the constant one). Thus, any two "composable 1-simplices" of $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$ admit *some* 2-simplex extending them. However, in the abstract simplicial set $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$, it is no longer possible to tell which 2-simplices arose from "strict composition" and which 2-simplices arose from "homotopy-coherent composition". And indeed, this is the entire point: *any* of the possible 2-simplex extensions of our two composable 1-simplices should be considered to be "just as good" as any other. In other words, the strict composition of composable 1-simplices in this simplicial set is *not even well-defined*.

The homotopy-coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet$ is the canonical example of a *quasicategory*. This is nothing other than a simplicial set $\mathtt{C}$ in which, for all $n \geq 2$, any string of n composable 1-simplices admits *some* extension to an n-simplex. If this string is selected by a morphism

$$\left(\Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \right) \to \mathtt{C}$$

of simplicial sets, then such an n-simplex $\Delta^n \to \mathtt{C}$ may be thought of as a witness to the fact that its 1-subsimplex

$$\Delta^{\{0,n\}} \to \Delta^n \to \mathtt{C}$$

is a composite of the string. Of course, in general such an extension will not be unique: indeed, all such extensions (for all strings of all lengths) will be unique precisely when $\mathtt{C}$ is the nerve of an ordinary category. Nevertheless, there is a strong sense in which such an extension is "essentially unique": the set of extensions of a string naturally extends to a *simplicial* set, which will always be contractible *when considered as a space* (i.e. as an object of the ∞-category $\mathcal{S} \simeq s\mathcal{S}\mathrm{et}[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!]$). Quasicategories are the fibrant objects of the *Joyal model structure* on the category $s\mathcal{S}\mathrm{et}$, to which the homotopy-coherent nerve functor defines a right Quillen equivalence

$$s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \leftarrow (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}} : \mathrm{N}^{\mathrm{hc}}(-)_\bullet.$$

Of course, the ∞-category of spaces is a rather abstract object. By contrast, its objects can be presented by topological spaces or by simplicial sets, both of which notions are quite concrete. For instance, one can speak of the "underlying set" of a topological space, whereas a *space* admits no such notion: a weak equivalence between topological spaces will not generally respect their underlying sets.

It would therefore appear to afford much more control to work directly with topologically- or simplicially-enriched categories, rather than considering them only as being enriched in the ∞-category of spaces (e.g. so that one can speak of the "underlying set" of a hom-space). Thus, the idea that an ∞-category should only be considered as being enriched over spaces runs directly against intuition, and against deeply-ingrained human urges for control.

However, ***the sheer power of this idea is impossible to overstate***.

To illustrate this striking phenomenon, we give two examples. For concreteness, both will concern the relationship between the 1-category $\mathcal{T}\mathrm{op}$ of topological spaces and the ∞-category $\mathcal{S}$ of spaces. For the present purposes, it will be convenient to consider the ∞-category of spaces as being presented by the homotopy-coherent nerve of the *topologically*-enriched category of CW complexes (although we only really consider it as a quasicategory at all to emphasize the non-strictness of its composition).[15,16]

Our first example of the power of ∞-categorical thinking illustrates the following paradigm: *working ∞-categorically, it's impossible to say the wrong thing.*

[15]Both functors in the adjunction $|-| : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}(-)_\bullet$ preserve finite products; applying them "locally" (i.e. to each hom-object individually) therefore defines an adjunction $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \rightleftarrows \mathcal{C}\mathrm{at}_{\mathcal{T}\mathrm{op}}$, and one can define the homotopy-coherent nerve of a topologically-enriched category simply by precomposing with its right adjoint.

[16]There is a notion of a model category being *compatibly enriched* over a given monoidal model category (the definition of which itself requires certain compatibilities between the model structure and the monoidal structure); for instance, both $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$ and $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ are compatibly self-enriched. Given a model category $\underline{\mathcal{M}}$ which is compatibly enriched over either $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$ or $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ and writing $\mathcal{M}$ for its underlying unenriched model category, the underlying ∞-category $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$ is presented by the $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$- or $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$-enriched category $\underline{\mathcal{M}}^{cf}$ of *bifibrant* (i.e. cofibrant and fibrant) objects. In particular, if either $x \in \mathcal{M}$ is not cofibrant or $y \in \mathcal{M}$ is not fibrant, then the enriched hom-object $\underline{\mathrm{hom}}_{\mathcal{M}}(x, y)$ will not generally have the "correct" weak equivalence class. In $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$, all objects are fibrant and CW complexes are cofibrant. In fact, they are not *all* of the cofibrant objects (these are "cell complexes and retracts thereof"), but their full inclusion into the topologically-enriched category $\underline{\mathcal{T}\mathrm{op}}^{cf}_{\mathrm{QS}}$ is a weak equivalence (and hence presents an equivalence of ∞-categories), so we've just restricted to them for simplicity of terminology.

Given a based CW complex X, its *suspension* is defined to be the pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

with itself of the inclusion of X into the cone $CX = (X \times [0,1])/(X \times \{1\})$ (as the subspace $X \times \{0\}$). This is an extremely useful construction in homotopy theory: for instance, it participates in a *suspension isomorphism*

$$\tilde{H}_i(X) \cong \tilde{H}_{i+i}(\Sigma X)$$

in (reduced) homology.

However, this definition itself is clearly not the "true" thing. After all, the cone CX is contractible, and indeed any other two contractible CW complexes into which X maps as closed inclusions would function just as well: more precisely, the resulting pushout would be weakly equivalent to the suspension ΣX. One gets the distinct sense that this "wants to be" the pushout

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma X \end{array}$$

along the unique terminal maps into (what end up being) the two cone points of ΣX – the only problem being that this diagram of topological spaces simply doesn't commute, let alone define a pushout.

On the other hand, this *canonically* defines a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & \searrow^{\simeq}_{\simeq} & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma X \end{array}$$

in the ∞-category of spaces! First of all, the map $X \to \Sigma X$ is given by the equatorial inclusion. Then, the homotopy-commutativity of each of the two triangles is selected by the canonical homotopy

$$X \times [0,1] \to CX$$

given by the formula

$$(x,t) \mapsto (x,t).$$

That is, this postcomposes to define homotopies

$$X \times [0,1] \to CX \to \Sigma X,$$

which select canonical paths in the hom-topological space $\underline{\hom}_{\mathcal{T}\mathrm{op}}(X, \Sigma X)$ between the equatorial inclusion and the inclusion of one or the other cone point.

Even better, this commutative square is a pushout in the ∞-categorical sense. Working ∞-categorically, the universal property of a pushout

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \coprod_A C \end{array}$$

has no choice but to read: "an object which, when mapped into a test object Y, corepresents the data of a map $B \to Y$, a map $C \to Y$, and a *path* witnessing the agreement of the two composites $A \to B \to Y$ and $A \to C \to Y$". Returning to our original example, we see that this is precisely the functor that the suspension ΣX was born to corepresent all along.

Finally, we reach our "impossible to be wrong" paradigm: for *any* contractible CW complexes B and C and *any* pair of maps $X \to B$ and $X \to C$, the ∞-categorical pushout

$$\begin{array}{ccc} X & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \coprod_X C \end{array}$$

is canonically equivalent (in the ∞-category of spaces) to the suspension ΣX. This is in stark contrast to the situation in the 1-category $\mathcal{T}\mathrm{op}$ of topological spaces, where one must demand that the maps $X \to B$ and $X \to C$ be closed inclusions. From this point of view, we learn the additional lesson that *working 1-categorically makes us want to force something which is* naturally *homotopy-coherent to be* unnaturally *strict.*

A pushout among CW complexes in which the two maps are closed inclusions is an example of a *homotopy pushout* in the model category $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$, which is in turn a particular example of a *homotopy colimit.* The theory of homotopy colimits in general model categories is well-studied, but it is fairly subtle and unreasonably technical: for instance, a homotopy colimit in a model category $\mathcal{M}$ over an indexing category $\mathcal{I}$ should be the left derived functor of the colimit functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{I}, \mathcal{M}) \to \mathcal{M},$$

but the requisite model structure needed to actually obtain this (i.e. a model structure on $\mathrm{Fun}(\mathcal{I}, \mathcal{M})$ for which this is a left Quillen functor) need not even exist. But more importantly, even in the extremely simple case of homotopy pushouts, these point-set considerations obscure the true and essential ∞-categorical meaning of the suspension construction $X \mapsto \Sigma X$, which – tying everything together – actually gives a conceptual explanation for the suspension isomorphism in the first place.

Our second example of the power of ∞-categorical thinking illustrates the following paradigm: *homotopy-coherence appears everywhere, and working ∞-categorically sweeps homotopy-coherence into the ambient machinery.*

Given a based topological space $X = (X, x)$, its *based loopspace* is the topological space

$$\Omega X = \{\gamma : [0,1] \to X \ : \ \gamma(0) = \gamma(1) = x\},$$

or equivalently the topological space $\underline{\hom}_{\mathcal{T}\mathrm{op}_*}(S^1, X)$ of based maps from the circle

$$S^1 = [0,1]/(0\sim 1)$$

into X.[17] By adjunction, there is a natural isomorphism

$$\pi_0(\Omega X) \cong \pi_1(X)$$

between the set of path components of ΩX and the fundamental group of X. Moreover, the group structure on the fundamental group $\pi_1(X)$ comes from concatenation of (homotopy classes of) based loops. For instance, given two based loops $\gamma_1, \gamma_2 \in \Omega X$, we can define a new based loop $(\gamma_1 * \gamma_2) \in \Omega X$ to be given by the formula

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

However, this formula is just the most straightforward option: any choice of "pinch map"

$$S^1 \xrightarrow{\Delta} S^1 \vee S^1$$

gives rise to a concatenation operation

$$\Omega X \times \Omega X \xrightarrow{\mu} \Omega X$$

which defines the same group structure on the set $\pi_0(\Omega X)$ of path components.

These considerations strongly suggest that the based loopspace ΩX should *itself* be some manner of "group", in which the multiplication law is given by concatenation of loops. However, a moment's reflection reveals that it is impossible to make this concatenation operation strictly associative, no matter which pinch map we choose.

On the other hand, in a sense, this failure of associativity is not so severe. Suppose that we fix a pinch map Δ on S^1 inducing a multiplication map μ on

[17] In fact, this is a completely dual object to the suspension ΣX: it's the ∞-*categorical pullback* of the diagram $\{x\} \to X \leftarrow \{x\}$.

ΩX. Then, the associativity diagram

$$\begin{array}{ccc} (\Omega X)^{\times 3} & \xrightarrow{\mathrm{id}_{\Omega X}\times\mu} & (\Omega X)^{\times 2} \\ {\scriptstyle \mu\times\mathrm{id}_{\Omega X}}\downarrow & & \downarrow{\scriptstyle \mu} \\ (\Omega X)^{\times 2} & \xrightarrow[\mu]{} & \Omega X \end{array}$$

does not strictly commute, but it commutes *up to homotopy*: in order to specify such a homotopy, it suffices to choose once and for all a homotopy witnessing the homotopy-commutativity of the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\Delta} & S^1 \vee S^1 \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{S^1}\vee\Delta} \\ S^1 \vee S^1 & \xrightarrow[\Delta\vee\mathrm{id}_{S^1}]{} & S^1 \vee S^1 \vee S^1, \end{array}$$

and this can be done straightforwardly (in essentially the same manner that one proves that the fundamental group is associative – it can be slightly easier to visualize the analogous picture with intervals instead of wedges of circles).

This is concordant with the ***core philosophy of higher category theory***: rather than merely positing the *existence* of a homotopy witnessing the homotopy-commutativity of the associativity diagram, we should instead keep track of such a homotopy as *additional data*.

These observations are sufficient for producing the group structure on $\pi_0(\Omega X)$, but they do not yet allow us treat ΩX as a "group" itself. For instance, suppose that we would like to concatenate *four* loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Omega X$. So far, we have only chosen a multiplication

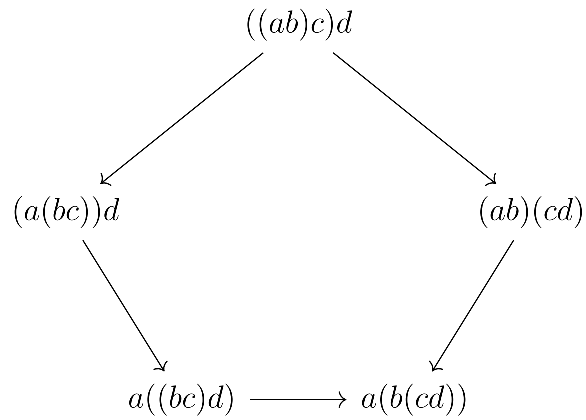
$$\mu : \mathrm{pt} \to \underline{\mathrm{hom}}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 2}, \Omega X)$$

along with an "associator", i.e. a path

$$\mu_3 : [0,1] \to \underline{\mathrm{hom}}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 3}, \Omega X)$$

between the two resulting composites $\mu \circ (\mathrm{id}_{\Omega X} \times \mu)$ and $\mu \circ (\mu \times \mathrm{id}_{\Omega X})$. As it turns out, μ determines five iterated multiplication maps $(\Omega X)^{\times 4} \to \Omega X$, which are in turn related by application of the associator at various stages:

these can be schematically organized into the famous "Mac Lane pentagon"



in which the arrows indicate "front-to-back" associations $(xy)z \to x(yz)$.[18] This can in turn be organized as a map from the boundary of a pentagon (thought of as a 1-dimensional simplicial complex) into the enriched hom-topological space

$$\underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 4}, \Omega X).$$

As this may a priori select a nontrivial loop, it is clear that we cannot yet declare our multiplication μ to be "unambiguously associative up to homotopy".

In his thesis [Sta61], Stasheff uncovered a strong sense in which the multiplication μ on ΩX is indeed "unambiguously associative up to homotopy", using what are now called the *Stasheff associahedra*. This is a sequence of convex polytopes $\{(\mathbb{A}_\infty)_n\}_{n\geq 2}$: it begins with $(\mathbb{A}_\infty)_2 = \mathrm{pt}$ and $(\mathbb{A}_\infty)_3 = [0,1]$, and so far we have observed maps

$$\mu_2 : (\mathbb{A}_\infty)_2 \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 2}, X)$$

and

$$\mu_3 : (\mathbb{A}_\infty)_3 \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 3}, X),$$

where the value of μ_3 on the boundary of $(\mathbb{A}_\infty)_3$ is determined by μ_2. Moreover, $(\mathbb{A}_\infty)_4$ is precisely the (filled-in) pentagon we have seen above, and we can similarly choose a map

$$\mu_4 : (\mathbb{A}_\infty)_4 \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 4}, X)$$

(which can likewise be determined "universally" by studying the pinch map Δ and its iterates) which extends the map on the boundary of $(\mathbb{A}_\infty)_4$ determined by μ_2 and μ_3. As $(\mathbb{A}_\infty)_4$ is convex and in particular contractible, this gives a precise sense in which *all four-fold multiplications are equivalent, up to contractible ambiguity*. Of course, this pattern continues: the maps $\mu_2, \ldots, \mu_{n-1}$

[18] This diagram appeared in Mac Lane's foundational study of monoidal categories [ML63], hence the name.

determine a map from the boundary of $(\mathbb{A}_\infty)_n$ into $\underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times n}, \Omega X)$, and it is possible to (universally) choose an extension over this contractible topological space. The relationships between these polytopes which inductively determine the maps on their boundaries assemble into certain structure maps which makes them into an *operad* (namely the $\mathbb{A}_\infty$ operad), and the compatible sequence of maps

$$\{\mu_n : (\mathbb{A}_\infty)_n \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times n}, \Omega X)\}$$

makes the topological space ΩX into an *algebra* over this operad.[19,20]

In fact, not only does the based loopspace of a based topological space carry the structure of an $\mathbb{A}_\infty$ algebra, but in a sense this structure *characterizes* based loopspaces: after we restrict to connected based spaces for obvious reasons, the based loopspace functor

$$\Omega : \mathcal{T}\mathrm{op}_*^{\geq 1} \to \mathcal{T}\mathrm{op}_*$$

defines an equivalence

$$\Omega : \mathcal{T}\mathrm{op}_*^{\geq 1}[\mathbf{W}^{-1}_{\text{w.h.e.}}] \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{A}_\infty}(\mathcal{T}\mathrm{op}_*)[\mathbf{W}^{-1}_{\text{w.h.e.}}]$$

onto the homotopy category of *grouplike* $\mathbb{A}_\infty$ topological spaces, i.e. those $\mathbb{A}_\infty$ algebras $Y \in \mathcal{T}\mathrm{op}_*$ for which the induced multiplication on $\pi_0(Y)$ makes it into a group (instead of just a monoid). In other words, a grouplike $\mathbb{A}_\infty$ structure on a topological space allows us to construct a *delooping* of that topological space (up to weak homotopy equivalence). The analogous result is false for "grouplike h-spaces", i.e. group objects in $\mathcal{T}\mathrm{op}[\mathbf{W}^{-1}_{\text{w.h.e.}}]$, for which we are only assured the *existence* of a homotopy making the associativity diagram commute (in the homotopy category). Thus, it is indeed only by *keeping track* of the homotopies making the (higher) associativity diagrams commute that we can construct a delooping.

Now, the $\mathbb{A}_\infty$ operad is an example of a ("non-symmetric") operad in topological spaces. Another example of an object in this category is the *associative operad*, denoted $\mathrm{Ass} \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$. This object is much simpler than $\mathbb{A}_\infty$: for all $n \geq 0$, we simply have $\mathrm{Ass}_n = \mathrm{pt}$. In other words, Ass parametrizes *strictly associative* multiplications.

[19] Actually, we have only parametrized n-fold multiplications for $n \geq 2$, whereas operads begin in degree 0. To extend this to a true $\mathbb{A}_\infty$ algebra structure, we should additionally specify the map $(\mathbb{A}_\infty)_0 = \mathrm{pt} \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 0}, \Omega X) \cong \Omega X$ selecting the basepoint (which functions as the "identity element" for the multiplication) as well as the map $(\mathbb{A}_\infty)_1 = \mathrm{pt} \to \underline{\hom}_{\mathcal{T}\mathrm{op}}((\Omega X)^{\times 1}, \Omega X)$ selecting the identity map on ΩX (the "1-fold multiplication").

[20] Operads were introduced by May in his landmark work [May72], in which he characterized *all* iterated loopspaces. As we will see presently, the $\mathbb{A}_\infty$ operad completely governs 1-fold loopspaces; this is also called the $\mathbb{E}_1$ operad, and more generally the $\mathbb{E}_n$ operad completely governs n-fold loopspaces for all n (including $n = \infty$, in a suitable sense).

In fact, the category $\mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$ of these *itself* admits a model structure, in which the weak equivalences are determined "level by level" (in $\mathcal{T}\mathrm{op}_{\mathrm{QS}}$). Moreover, Ass is the terminal object of $\mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$, and the unique map $\mathbb{A}_\infty \to$ Ass is a cofibrant replacement (and in particular, a weak equivalence).

This is relevant for the following reason. First of all, any object $Y \in \mathcal{T}\mathrm{op}$ determines an *endomorphism operad*

$$\mathcal{E}\mathrm{nd}^{\mathrm{ns}}(Y) \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$$

given by

$$\mathcal{E}\mathrm{nd}^{\mathrm{ns}}(Y)_n = \underline{\mathrm{hom}}_{\mathcal{T}\mathrm{op}}(Y^{\times n}, Y).$$

Moreover, as suggested by the above discussion, for an arbitrary operad $\mathcal{O} \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$, one can *define* an $\mathcal{O}$-algebra structure on Y to be a morphism

$$\mathcal{O} \to \mathcal{E}\mathrm{nd}^{\mathrm{ns}}(Y)$$

of operads. Thus, to say that $\mathbb{A}_\infty$ is "good for mapping out of" (in a way that Ass is not) is to say that certain topological spaces (e.g. and i.e. based loopspaces) "want" to be associative algebras, but are in fact only $\mathbb{A}_\infty$ algebras.

In fact, the term "$\mathbb{A}_\infty$ operad" has come to refer to *any* cofibrant replacement of the associative operad. Moreover, a weak equivalence between cofibrant operads induces a Quillen equivalence between their model categories of algebras. Thus, we see that the point-set $\mathbb{A}_\infty$ operad in topological spaces is not the "true" thing: the homotopy category of based loopspaces can be organized as the homotopy category of grouplike algebras over *any* cofibrant replacement of the associative operad.

By now, the punch line should be clear: based loopspaces *are* associative algebras, but only when considered in the ∞-*category* of spaces! Moreover, the above equivalence of homotopy categories lifts to an equivalence

$$\Omega : \mathcal{S}_*^{\geq 1} \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathrm{Ass}}(\mathcal{S}) = \mathcal{G}\mathrm{rp}(\mathcal{S})$$

of ∞-categories. Thus, as advertised, the homotopy-coherence inherent in the very foundations of ∞-categories turns a complicated and un-"true" assertion about not-even-canonical point-set operads into the simple, canonical, and compelling statement that we were really after all along: based loopspaces of pointed spaces determine group objects in spaces.[21]

On the other hand, there is another approach to studying the homotopy category of based loopspaces: in fact, it turns out that the canonical map $\mathbb{A}_\infty \to$ Ass *also* induces an equivalence

$$\mathrm{Alg}^{\mathrm{gp}}_{\mathrm{Ass}}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}] \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{A}_\infty}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}]$$

[21]The fact that this induces an equivalence when we restrict to connected based spaces is a homotopical form of *Koszul duality*, which features prominently in the study of deformation theory.

of homotopy categories (even though $\mathrm{Ass} \in \mathrm{Op}^{\mathrm{ns}}(\mathcal{T}\mathrm{op})$ is not cofibrant). In particular, any grouplike $\mathbb{A}_\infty$ algebra in $\mathcal{T}\mathrm{op}$ is weakly equivalent (as an $\mathbb{A}_\infty$ algebra) to a *topological group*. Thus, one can also study the homotopy category of based loopspaces by studying the homotopy category of topological groups.

However, it is only due to the simplicity of the $\mathbb{A}_\infty$ operad that such strictification is possible. For instance, a cofibrant replacement of the *commutative operad*

$$\mathrm{Comm} \in \mathrm{Op}(\mathcal{T}\mathrm{op})$$

(which can only be defined as a "symmetric" operad – in fact, it is likewise the terminal object of $\mathrm{Op}(\mathcal{T}\mathrm{op})$) is called an "$\mathbb{E}_\infty$ operad", and restriction along the canonical map $\mathbb{E}_\infty \to \mathrm{Comm}$ determines a functor

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{T}\mathrm{op}) \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{T}\mathrm{op})$$

which does *not* induce an equivalence on homotopy categories. In particular, the induced functor

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}] \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{T}\mathrm{op})[\mathbf{W}^{-1}_{\mathrm{w.h.e.}}]$$

on homotopy categories is not essentially surjective: not every topological space equipped with a homotopy-coherently commutative and associative multiplication can be rigidified to one with a strictly commutative and associative multiplication.

From a more philosophical perspective, we posit that it should feel *morally reprehensible* to attempt to force a based loopspace to be something which it is not: it is truly and essentially a homotopy-coherent object, and its strictifiability is ultimately just an intriguing coincidence.

In fact, recall from §2 that a one-object category is completely specified by the monoid of endomorphisms of its unique object. In an identical fashion, a one-object topologically-enriched category is completely specified by the topological monoid of endomorphisms of its unique object. Thus, the coincidence that based loopspaces can be rectified to topological groups is (up to questions of grouplikeness) nothing other than a "one-object" version of the coincidence that topologically-enriched categories present ∞-categories! We may therefore view this connection as justifying our philosophical assertion that *we never should have considered ∞-categories as having strictly associative composition in the first place.*[22]

[22] Recall that before extolling the philosophical advantages of homotopy-coherent models for ∞-categories (over strict ones), we actually began this section by mentioning certain technical advantages that they also enjoy. In fact, it turns out that these technical advantages can *themselves* be seen as arising from the fact ∞-categories fundamentally "want" to be homotopy-coherent objects. Thus, these technical and philosophical advantages are actually two sides of the same coin.

Of course, these two examples of the power of ∞-categorical thinking are merely toys, which we chose in order to highlight the differences between working strictly and working homotopy-coherently. The real fun begins when one actually starts to *use* ∞-category theory, at which point the world becomes a magical place: one's power to make new definitions is limited only by one's imagination, and one's ability to prove new theorems is limited only by the clarity of one's understanding (at least as far as the purely formal aspects are concerned). The many fussy details that arise when one attempts to use point-set techniques to work homotopy-coherently simply melt away: they were in fact irrelevant all along to the true and underlying mathematics, and their disappearance into the ambient machinery brings with it a harmony that is only possible when intuition and language are once again aligned. Thus, paradoxically, by *discarding* such emotional crutches as underlying sets and strict composition and by *embracing* the apparent chaos and uncontrol of homotopy-coherence, we acquire a measure of power of which previous generations of mathematicians could barely have dreamed.

Appendix A. The praxis of ∞-categories

In case it was not evident from the discussion of §5, we now make an explicit clarification: in reality, a large number of users of ∞-categories throughout mathematics do not actually choose *any* particular model category of them, instead working in a purely formal manner and only making reference to universal constructions (such as limits, colimits, adjoint functors, etc.).

Most pragmatically, this (absence of) choice can be justified by declaring that such manipulations are "secretly" taking place among quasicategories. Indeed, although quasicategories are in the end nothing more than certain simplicial sets, they collectively assemble into a quasicategory *of quasicategories*, in which e.g. it is only possible to speak of homotopy-coherent composition of functors between them. Moreover, the theory of quasicategories has been developed extensively, most notably by Joyal and Lurie. As a result, nearly any 1-categorical maneuver one might wish to imitate (e.g. an appeal to the

In order to see this, recall that the technical disadvantages e.g. of simplicially-enriched categories are ultimately due to the failure of the cartesian product of two cofibrant objects to again be cofibrant. Indeed, this failure is in turn due to the fact that the "correct" hom-set must encode all *homotopy-coherent* functors. If the target object already accounts for this homotopy-coherence (as does e.g. a quasicategory), then the source object doesn't need to (and indeed, all objects of $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ are cofibrant). But if the target object is forced to be strict (as is e.g. a simplicially-enriched category), then to get the correct hom-set we need to account for our desired homotopy-coherence in the source. As taking a product generally introduces new composites that weren't present in either factor individually (e.g. consider the product $[1] \times [1]$), it should come as no surprise that products of cofibrant simplicially-enriched categories don't generally remain cofibrant.

adjoint functor theorem) can be rigorously performed in the quasicategorical setting.[23]

The "underlying ∞-category" of this quasicategory – or indeed, of e.g. either relative category $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ or $(\mathfrak{cat}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ – is denoted $\mathcal{C}\mathrm{at}_\infty$ and is referred to as *the ∞-category of ∞-categories*.

Appendix B. Model categories and ∞-categories

A technically advantageous model category of ∞-categories is absolutely essential for the full and rigorous development of the theory of ∞-categories. Thus, the theory of ∞-categories rests firmly on the theory of model categories.

However, both can be used as frameworks for abstract homotopy theory. On the one hand, a model structure on a relative category $(\mathcal{M}, \mathbf{W}) \in \mathfrak{relcat}$ provides an efficient method of making computations not just in its localization

$$\mathcal{M}[\mathbf{W}^{-1}] \in \mathfrak{cat}$$

but in its *derived localization*

$$\mathcal{M}[\![\mathbf{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_\infty$$

(which is indeed its localization when considered as a *relative ∞-category*). On the other hand, essentially every ∞-category of lasting interest can be presented by a model category $\mathcal{M}$ in this sense. It is therefore often analogized that model categories are to ∞-categories as atlases are to manifolds: a model category is a convenient presentation of an ∞-category, but not every operation that one might like to perform in an ∞-category can be presented within a given model category.[24]

By no means does the theory of ∞-categories render the theory of model categories obsolete, even beyond the obvious issue of logical reliance. To wit, model categories are still an indispensable component of the homotopical toolkit because *it is essentially impossible to perform any non-formal computations using ∞-category theory alone.*

To give an example of this, we return to the original thread with which our story began. Given an abelian category $\mathcal{A}$, the relative category $(\mathrm{Ch}(\mathcal{A}), \mathbf{W}_{\mathrm{q.i.}})$ is the natural home of "resolutions" of objects of $\mathcal{A}$. Out of this, we can form

[23] For a beautiful and compelling introduction to quasicategories, we refer the interested reader to [Lur09, Chapter 1].

[24] However, the analogy breaks down quickly: for example, the existence of a model category presenting an ∞-category implies the existence of all limits and colimits in the latter (or at least the finite ones, depending on which variant of the definition "model category" one chooses). As a result, not every ∞-category can be presented by a model category.

the *derived ∞-category* of $\mathcal{A}$, namely the ∞-categorical localization

$$\mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\mathrm{q.i.}}^{-1}]\!] \in \mathcal{C}\mathrm{at}_\infty$$

of this relative category. This admits a canonical functor

$$\mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\mathrm{q.i.}}^{-1}]\!] \to \mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\mathrm{q.i.}}^{-1}],$$

which witnesses the ordinary localization $\mathrm{Ch}(\mathcal{A})[\mathbf{W}_{\mathrm{q.i.}}]$ as the *homotopy category* of the ∞-categorical localization (obtained by applying the functor $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$ "locally" (i.e. to each hom-space individually)); the derived ∞-category of $\mathcal{A}$ is thus a *refinement* of the ordinary derived category, and we will henceforth reappropriate the notation

$$\mathscr{D}(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})[\![\mathbf{W}_{\mathrm{q.i.}}^{-1}]\!]$$

accordingly.

Now, suppose we are given two objects $M, N \in \mathcal{A}$, and suppose we would like to understand the hom-space

$$\hom_{\mathscr{D}(A)}(M, N).$$

Though it arises from a modern construction, this space is often of classical interest: for instance, if $\mathcal{A} = \mathrm{Mod}_R$, then its homotopy groups are precisely the Ext groups $\mathrm{Ext}^*_R(M, N)$. However, we are once again faced with precisely the same issue that we confronted in §2: the derived ∞-category admits a universal characterization *as an ∞-category*, but this abstract characterization takes place at the wrong "category-level" for direct computation within it to be even remotely possible. Rather, it remains as necessary as ever to take *resolutions*, i.e. to make use of a model structure on the relative category $(\mathrm{Ch}(\mathcal{A}), \mathbf{W}_{\mathrm{q.i.}})$. For instance, if $\mathcal{A} = \mathrm{Mod}_R$, then it is necessary to take either a projective resolution of M or an injective resolution of N.[25]

On the other hand, ∞-categories make possible a number of obviously desirable maneuvers which model categories do not accommodate (or do not easily accommodate). The consideration of *functors* is surely the most important example.

Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, it is utterly straightforward to define the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ (whose morphisms are natural transformations). For example, if $\mathtt{C}$ and $\mathtt{D}$ are quasicategories which respectively present $\mathcal{C}$ and $\mathcal{D}$, then the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presented by

[25] On the other hand, it must also be said that many operations in the literature which happen to be performed within model categories are actually essentially formal and hence could be done equally well – or perhaps better, in the interest of conceptual clarity – in their underlying ∞-categories.

the internal hom-object $\underline{\hom}_{s\mathcal{S}\mathrm{et}}(\mathtt{C},\mathtt{D})$ in simplicial sets. As an ∞-category, this represents the desired functor

$$\mathcal{E} \mapsto \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{E}, \mathrm{Fun}(\mathcal{C},\mathcal{D})) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{E}\times\mathcal{C},\mathcal{D});$$

there's nothing more to it.

By contrast, almost without exception the only meaningful "morphisms" between model categories are given by Quillen adjunctions. Moreover, a Quillen adjunction

$$\mathcal{M} \rightleftarrows \mathcal{N}$$

between model categories induces not just a derived adjunction $\mathcal{M}[\mathbf{W}^{-1}] \rightleftarrows \mathcal{N}[\mathbf{W}^{-1}]$ (as described in Theorem 2.2) but an ∞-categorical adjunction

$$\mathcal{M}[\![\mathbf{W}^{-1}]\!] \rightleftarrows \mathcal{N}[\![\mathbf{W}^{-1}]\!]$$

between underlying ∞-categories. Thus, model categories provide an acutely restrictive framework if one is interested in non-adjoint functors between underlying ∞-categories.

However, given a diagram category $\mathcal{I}$ and a model category $\mathcal{M}$, it is sometimes possible to endow the functor category $\mathrm{Fun}(\mathcal{I},\mathcal{M})$ with a "pointwise" model structure (i.e. one whose weak equivalences are precisely those natural transformations whose components are all weak equivalences in $\mathcal{M}$). For example, under certain (often-satisfied) restrictions on $\mathcal{M}$, there exists a *projective model structure* $\mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{proj}}$, while under certain (still often-satisfied) further restrictions on $\mathcal{M}$ there also exists an *injective model structure* $\mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{inj}}$. When they exist, these model structures can be used to compute homotopy co/limits, as they participate in Quillen adjunctions

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathcal{M} : \mathrm{const}$$

and

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{inj}} : \lim .$$

Alternatively, if $\mathcal{I}$ is a *Reedy* category (which condition is quite restrictive but is satisfied for a reasonably large class of examples of practical interest, including e.g. the categories $\mathbf{\Delta}$ and $\mathbf{\Delta}^{op}$), then for *any* model category $\mathcal{M}$ there exists a *Reedy model structure* $\mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{Reedy}}$. However, in general the Reedy model structure need not be compatible with either the colimit functor or the limit functor in the sense described above.

As should be clear from the complexity of this discussion, in practice these pointwise model structures can be a nuisance. For instance, there does not generally exist such a model structure on $\mathrm{Fun}(\mathcal{I},\mathcal{M})$ which is compatible with *both* the colimit functor and the limit functor, and so as a result one must pass through the Quillen equivalence

$$\mathrm{id} : \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{proj}} \rightleftarrows \mathrm{Fun}(\mathcal{I},\mathcal{M})_{\mathrm{inj}} : \mathrm{id}$$

to mediate between the opposite "handedness" of these two model structures. Moreover, this entire discussion only allows $\mathcal{J}$ to be a diagram *1-category*: it is extremely difficult to work with diagrams in a model category which are meant to present diagrams indexed by a more general ∞-category.

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