Kinds of K-theory

- Algebraic K-theory
- Topological K-theory
- Derived K-theory
- Motivic K-theory

Define a v.h.

- Continuous assignment of vector spaces to points.
- \( E \to B \), \( u \) a cover \( B \), \( \pi \) a map, \( U_a = U_a \times \mathbb{C}^n \), and

\[
\begin{align*}
C^n & \to U_a \times \mathbb{C}^n \\
& \to B \\
\end{align*}
\]

should be a linear isomorphism of vector spaces.

- Note that \( U_a \subset \mathbb{C}^n \), and if we understand \( U_a \subset \mathbb{A} \) then we can recover \( E \), a UCal-bundle can be thickened to a v.h. by \( \text{Ucal} \times \mathbb{C}^n \) fiberwise.

- \( T \to B \) \( \xrightarrow{\pi} \) Vector Spaces \( \xrightarrow{\text{Spaces}} \) Spaces

\[
\begin{align*}
\text{colimit} & \xrightarrow{\oplus} \text{Spaces} \\
C^n & \text{- bundle over } \oplus B.
\end{align*}
\]

The map \( E_0 \to E_\infty \) is defined by taking

\[
E_0 \star I \to E_\infty
\]
Maps $\text{Map}(X \times \mathbb{R}, \text{VectorSpaces})$ should be "like" continuous maps $X \to \text{VectorSpace}_1$. What is something like $\text{VectorSpace}_1$?

*Classifying spaces:* to a topological group $G$, group $G$-category $\cdots$ we can associate a space $BG$ so that $X \to BG$ is a $G$-bundle. The identity $BG \to BG$ gives a $G$-bundle $G \to EG \to BG$, and $EG$ is contractible, and uniquely determined by this property.

$S^1 \cong \mathbb{C}^\infty \to C^\infty$, for instance, is a model for $\text{EU}(1) \to B\text{U}(1)$.

Recall $C^\infty = C^\infty(\mathbb{R}, \mathbb{R}) \cong B\text{U}(1)$, $\text{EU}(1) \cong B\text{U}(1)$.

Similarly, $\exists B\text{U}(n)$ which classify rank $n$ vector bundles. Let's study $H^*\text{U}(n)$ and $H^*\text{U}(n)$, in $\text{U}(n) = H^*\text{U}(n)$.

**The Splitting principle:**

Every vector bundle $X \to \text{BU}(n)$ can be pulled back along $PC(V) \to X$ to a sum of line bundles, and $H^*X \cong H^*\text{U}(V)$.

$H^*C^\infty = H^*(C^0 \cup C^1 \cup C^2 \cup \cdots) = \{0, 0, 0, 0, 0, 0, \cdots\}$, or $H^*(C^\infty, H^*(S^1)) \to H^*(\text{EU}(1) \times \mathbb{R}) = \mathbb{Z}.$

$H^*C^\infty = \mathbb{Z}^{C_1}$.

$H^*(C^\infty, H^*(S^1)) \to H^*(\text{EU}(1) \times \mathbb{R}) = \mathbb{Z}.$
Maybe best to just say: Similar fiber sequence exist of the type

\[ S^{2n-1} \rightarrow \Omega S^n(n-1) \rightarrow BU(n), \]

and similar analysis with the SSS shows \( H^* BU(n) \cong \mathbb{Z}[c, \ldots, c_n]. \)

And, in the end, we can take a colimit to get \( H^* BU \cong \mathbb{Z}[c, c_2, \ldots]. \)

\[ \approx \text{Sym}(H^*(BU)). \]

But \( \Omega X \) be a bundle, and split to \( \mathbb{L} \rightarrow \mathbb{L} \rightarrow \mathbb{L} \rightarrow Y. \) Then we get a

sequence of cohomology classes \( c, \ldots, c_n \) in \( H^* \mathbb{X} \) determined by pulling back

\[ c_1, c_2, \ldots, c_n \in H^* \mathbb{X} \]

certain classes \( c, \ldots, c_n \) defined over \( H^*(BU) \)
determined by \( c_1(c_2^0) = c, \ c_n(c_0^0) = 1, \) and \( c(V \oplus U) = c(V) \cdot c(U). \)

A priori, these give cohomology classes in \( H^* Y \) by pullback, but these classes

secretly live in \( H^* BU(n). \)

The homology of \( BU. \)

Define \( BU = \lim_{n \rightarrow \infty} BU(n), \) the classifying space for "stable" vector bundles. This

gives an inverse --- he classifies stable equivalence classes of different bundles.

\( BG \) is the space of self-maps of frames

\[ n \text{ diml subspaces} \rightarrow \left\{ \text{inclusions} \right\} \rightarrow \left\{ \text{unitary maps} \right\} \rightarrow \left\{ \text{unitary maps} \rightarrow \left\{ \text{unitary maps} \right\} \rightarrow \left\{ \text{unitary maps} \right\} \right\} 

Each of these are simplicial objects.
Snaith's Theorem

So, $B^1 \Sigma BU \cong BU$, or $\Sigma^2 BU \cong BU$, or $\exists S^2 \to BU$ which we've found an inverse for -- this is $\sum_{L=1}^{L=52}$.

\[ \begin{array}{c}
\Sigma^3 \mathbb{C}P^\infty & \xrightarrow{[L=1]} & BU \\
\downarrow \cong & \downarrow \cong & \downarrow \cong \\
\Sigma^3 \mathbb{C}P^\infty & \cong & KU \\
\end{array} \]

It turns out that this is a homotopy equivalence of spectra. E. Geiser has given a new proof of this; I'd like to point out one part.

Assume $M = R \mathbb{C}P^\infty$, $R \mathbb{C}P^\infty$, $R$ is an even-periodic ring spectrum.

\[ 
A = \text{Sym } M \xrightarrow{\epsilon} \Sigma^1 I^R, R \xrightarrow{\cong} \\
I = \text{Aug } A \\
\text{Add}(BU, R) \\
\text{Add}(BU, R) \\
\]

\[ \begin{array}{c}
0 & \xrightarrow{\epsilon} & \Sigma^1 I^R & \xrightarrow{\epsilon} & \Sigma^1 I^R & \xrightarrow{\epsilon} & 0 \\
0 & \xrightarrow{\epsilon} & R^0 BU & \xrightarrow{\epsilon} & R^0 BU & \xrightarrow{\epsilon} & 0 \\
0 & \xrightarrow{\epsilon} & \Sigma^1 I^R & \xrightarrow{\epsilon} & \Sigma^1 I^R & \xrightarrow{\epsilon} & 0 \\
0 & \xrightarrow{\epsilon} & R^0 BU \times BU & \xrightarrow{\epsilon} & R^0 BU \times BU & \xrightarrow{\epsilon} & 0 \\
\end{array} \]

This lifts to an equivalence maps $(\Sigma^3 \mathbb{C}P^\infty, R) \to \text{maps}(KU, R)$.