All cohomology is with $\mathbb{Z}/2$ coefficients.

For any $a \in H^n(X)$, the external cup square $a \otimes a \in H^{2n}(X \times X)$ induces a map

$$f : X \times X \to K(\mathbb{Z}/2, 2n).$$

It can be shown that this map factors as

$$\begin{array}{ccc} X \times X & \xrightarrow{f} & K(\mathbb{Z}/2, 2n) \\ \downarrow & \nearrow_{g} & \\ (X \times X) \times_{\mathbb{Z}/2} E\mathbb{Z}/2, & & \end{array}$$

where $\mathbb{Z}/2$ acts on the product by permuting the factors and $E\mathbb{Z}/2$ can be taken to just be S^∞. If you unravel what this means, it says that our original map f was homotopic to the map obtained by first switching the coordinates and then applying f. It also says that this homotopy, when applied twice to get a homotopy from f to itself, is homotopic to the identity homotopy, and we similarly have a whole series of higher "coherence" homotopies. Now we have the diagonal map

$$\Delta : X \times B\mathbb{Z}/2 \to (X \times X) \times_{\mathbb{Z}/2} E\mathbb{Z}/2,$$

so we get a map

$$g\Delta : X \times B\mathbb{Z}/2 \to K(\mathbb{Z}/2, 2n).$$

But $H^*(B\mathbb{Z}/2) = \mathbb{Z}/2[t]$, so this gives a class $Sq(a) \in H^{2n}(X)[t]$. If we write $Sq(a) = \sum s(i)t^i$, it can be shown that $s(i) = Sq^{n-i}(a)$.

What does this mean? Well, if our map f actually *was* invariant under switching the factors (which you might think it ought to be, given that it appears to be defined symmetrically in the two factors), we could take g to just be the projection onto $X \times X$ followed by f. This would mean that $Sq(a)$ comes from just projecting away the $B\mathbb{Z}/2$ and then using a^2, i.e. $Sq^n(a) = a^2$ and $Sq^i(a) = 0$ for all other i. Thus the nonvanishing of the lower Steenrod squares somehow measures how the cup product, while *homotopy*-commutative (in terms of the induced maps to Eilenberg-MacLane spaces), cannot be straightened to be actually commutative. Indeed, in the universal example $X = K(\mathbb{Z}/2, n)$, then map f is exactly the universal map representing the cup product of two cohomology classes of degree n.

– *Eric Wofsey, accepted answer at*
`http://mathoverflow.net/questions/461/understanding-steenrod-squares`