The Stable Homotopy Category

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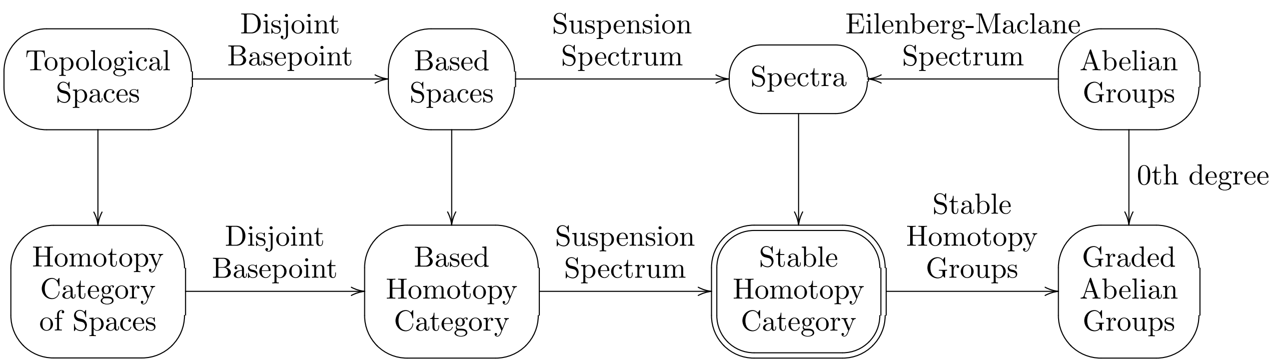
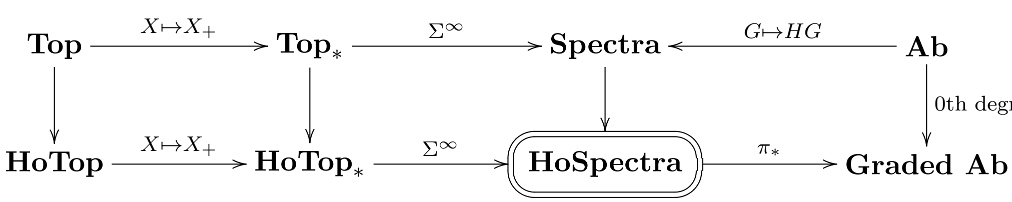
These goal of these notes is to explain what a spectrum is. There are many different definitions, and it is not obvious to a nonspecialist how they are equivalent. Therefore we will begin with a description of the *properties* that we want spectra to have, before actually *defining* them. We assume familiarity with homology, cohomology, and homotopy groups, along with categories, functors, and natural transformations.

To start, spectra should form a category, with functors coming in and going out to other categories that we care about. We can capture this in a commuting diagram of functors:



or in shorthand,



There are many different definitions of the category labelled **Spectra**, most of which are not equivalent. Each of them in turn gives a definition of the stable homotopy category **HoSpectra**, but here they are almost always equivalent. So we can prove many things using only abstract properties of the stable homotopy category **HoSpectra**, avoiding the "implementation details" found in **Spectra**.

These notes are an ongoing project, aimed at an elementary grad-student level. I greatly appreciate feedback. I'd like to thank Aaron Mazel-Gee for providing a lot, and Ilya Grigoriev for insisting that I put bubbles on the first page.

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1 Properties We Want the Stable Homotopy Category to Have

1.1 Topological Spaces

We care about the category of compactly generated weak Hausdorff (CGWH) topological spaces, which we'll denote **Top**. The *homotopy category of spaces* **HoTop** has the same objects as **Top**, but the morphisms from X to Y are homotopy classes of maps between CW approximations $[\widetilde{X}, \widetilde{Y}]$. By Whitehead's theorem, every weak homotopy equivalence in **Top** becomes an isomorphism in **HoTop**, and conversely, every map that becomes invertible in **HoTop** is a weak homotopy equivalence.

We could also start with the category of based spaces, denoted $\mathbf{Top}_*$. The *homotopy category of based spaces* $\mathbf{HoTop}_*$ has the same objects (the based CGWH spaces), and the morphisms are based homotopy classes of maps between based CW approximations $[\widetilde{X}, \widetilde{Y}]$. To pass from unbased spaces to based spaces, we add a disjoint basepoint.

Remark. If you know about model categories, we have described a category that is isomorphic to the usual one given by the Quillen model structure on $\mathbf{Top}$ and $\mathbf{Top}_*$. We construct the cofibrant replacement QX by factoring the weak equivalence $\widetilde{X} \longrightarrow X$ into a trivial cofibration $\widetilde{X} \longrightarrow QX$ and a fibration $QX \longrightarrow X$. The composition of cofibrations $\emptyset \longrightarrow \widetilde{X} \longrightarrow QX$ is a cofibration, and $QX \longrightarrow X$ is a weak equivalence by 2 out of 3. The usual construction of the factorization is natural and makes $\widetilde{X} \longrightarrow QX$ a strong homotopy equivalence, so it gives a natural bijection $[\widetilde{X}, \widetilde{Y}] \longleftrightarrow [QX, QY]$. Therefore our construction of $\mathbf{HoTop}$ and $\mathbf{HoTop}_*$ is isomorphic to the usual one.

1.2 Suspension and Abelian Groups

We claim that there is a category called **HoSpectra**, and a functor $\Sigma^\infty : \mathbf{HoTop}_* \longrightarrow \mathbf{HoSpectra}$ with the following properties:

- There is a suspension functor $\Sigma : \mathbf{HoSpectra} \longrightarrow \mathbf{HoSpectra}$ that agrees with the usual (reduced) suspension of based spaces:

$$\begin{array}{ccc}
\mathbf{Top}_* & \xrightarrow{\Sigma} & \mathbf{Top}_* \\
\downarrow & & \downarrow \\
\mathbf{HoTop}_* & \xrightarrow{\Sigma} & \mathbf{HoTop}_* \\
\downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\
\mathbf{HoSpectra}_* & \xrightarrow{\Sigma} & \mathbf{HoSpectra}_*
\end{array}$$

  Moreover, Σ is an equivalence of categories from **HoSpectra** to itself. So every object of **HoSpectra** is isomorphic to the suspension of some other object. This certainly wasn't true for $\mathbf{Top}_*$ or $\mathbf{HoTop}_*$.

- There is a loopspace functor $\Omega : \mathbf{HoSpectra} \longrightarrow \mathbf{HoSpectra}$ that agrees with the usual based loopspace:

$$
\begin{array}{ccc}
\mathbf{Top}_* & \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} & \mathbf{Top}_* \\
\downarrow & & \downarrow \\
\mathbf{HoTop}_* & \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} & \mathbf{HoTop}_* \\
\downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\
\mathbf{HoSpectra}_* & \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} & \mathbf{HoSpectra}_*
\end{array}
$$

In **HoSpectra**, the functors Σ and Ω are inverse equivalences, so $\Sigma \circ \Omega$ and $\Omega \circ \Sigma$ are naturally isomorphic to the identity. Every object is isomorphic to the loopspace of some other object. Again, this isn't true in $\mathbf{Top}_*$ or $\mathbf{HoTop}_*$.

- Given objects X and Y in **HoSpectra**, the set of morphisms $[X, Y]$ can be turned into an abelian group. Intuitively, we think of X as a suspension $\Sigma X'$, and we use the usual "pinching" and "flipping" constructions on $[S^n, Z] = \pi_n(Z)$ to add or negate maps $\Sigma X' \longrightarrow Y$. (There is also an analogue of the Eckmann-Hilton argument to show that addition in $[\Sigma^2 X'', Y]$ is commutative, and a natural bijection $[\Sigma X', Y] \cong [X', \Omega Y]$.) Composition of morphisms $[X, Y] \times [Y, Z] \longrightarrow [X, Z]$ is bilinear, so it induces a homomorphism of abelian groups $[X, Y] \otimes [Y, Z] \longrightarrow [X, Z]$.

- The category **HoSpectra** has coproducts (wedge sums) $X \vee Y$ and products $X \times Y$. There is a zero object $*$, coming from the one-point based space $*$ in $\mathbf{Top}_*$. This means that for every object X, there are unique maps $* \longrightarrow X \longrightarrow *$. This gives natural maps

$$
\begin{cases}
X \vee * \longrightarrow X \\
X \longrightarrow X \times * \\
X \vee Y \longrightarrow X \times Y
\end{cases}
$$

  The first two rows are always isomorphisms, using the data we gave above. In **HoSpectra**, the third map is also an isomorphism. This was not true for based spaces!

- The last two bullet points combine to tell us that **HoSpectra** is an *additive category.*

- We can extend the abelian group $[X, Y]$ into a graded abelian group $[X, Y]_*$, containing $[X, Y]$ as the 0th level. We simply define $[X, Y]_n = [\Sigma^n X, Y]$. Notice that n can be any integer, since suspension Σ has an inverse equivalence Σ^{-1}. Alternatively, we could define $[X, Y]_{-n} = [X, \Sigma^n Y]$.

All of these properties are completely analogous to the basic properties of graded abelian groups. Suspension is the operation that shifts the grading by one. Looping shifts the grading by one in the opposite direction. If G and H are two graded abelian groups, the set of graded homomorphisms

$\{G_i \longrightarrow H_i\}_i$ between them forms an abelian group. This can be extended to a graded abelian group of "shifted homomorphisms" $\{G_i \longrightarrow H_{i+n}\}_i$. The coproduct $G \oplus H$ and the product $G \times H$ are naturally isomorphic. Many of these properties also carry over to the simpler setting of ungraded abelian groups.

Before we continue the analogy with abelian groups, let's list a few more topological properties of **HoSpectra**:

- Define the *sphere spectrum* to be $\mathbb{S} = \Sigma^\infty S^0$. Given an object X in the stable homotopy category, we define its *stable homotopy groups* to be

  $$\pi_n(X) = [\mathbb{S}, X]_n = [\Sigma^n \mathbb{S}, X]$$

  Again, notice that n can be a negative integer and this still makes sense. If K is a based space, then we require that $\pi_n(\Sigma^\infty K)$ be naturally isomorphic to the usual stable homotopy groups

  $$\begin{aligned} \pi_n^S(K) &:= \operatorname{colim}_{k\to\infty} \pi_{k+n}(\Sigma^k K) \\ &= \operatorname{colim}_{k\to\infty} \pi_n(\Omega^k \Sigma^k K) \\ &= \pi_n(\Omega^\infty \Sigma^\infty K) \end{aligned}$$

  Notice that $\pi_n(\Sigma^\infty K)$ is zero for negative n.

- The functor Σ^∞ has a right adjoint $\Omega^\infty : \mathbf{HoSpectra} \longrightarrow \mathbf{HoTop}_*$. This means that for a based space K and a spectrum X,

  $$[\Sigma^\infty K, X] \cong [K, \Omega^\infty X]$$

  In particular, for nonnegative n this gives $\pi_n(X) \cong \pi_n(\Omega^\infty X)$. Of course, $\Omega^\infty X$ is a space, so it has no negative homotopy groups.

- The objects X in **HoSpectra** whose homotopy groups $\pi_n(X)$ vanish for negative n are called *connective spectra*. By the above, Σ^∞ takes every based space to a connective spectrum.

- Whitehead's Theorem: If a map $f : X \longrightarrow Y$ in **HoSpectra** induces an isomorphism $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$, then f is an isomorphism.

1.3 Tensor Products and Rings

Carrying the analogy with abelian groups even further, we can define a tensor product on objects of **HoSpectra**. Before describing its properties, let's recall the basic properties of the tensor product $\otimes = \otimes_{\mathbb{Z}}$ of abelian groups:

$$\otimes : \mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$$

Here $\mathbf{Ab} \times \mathbf{Ab}$ is a "product category," whose objects are pairs of abelian groups and morphisms are pairs of morphisms. So we can think of the tensor product as an operation on abelian groups that is a "functor in each slot."

Let's follow the convention that $\mathbf{Ab}(A, B)$ is the set of linear maps from A to B, and $\mathrm{Hom}(A, B)$ is the abelian group of linear maps. So if we forget that $\mathrm{Hom}(A, B)$ is a group, we get the set $\mathbf{Ab}(A, B)$. The defining property of $\otimes$ is that linear maps $A \otimes B \longrightarrow C$ correspond naturally to bilinear maps $A \times B \longrightarrow C$. A bilinear map is the same thing as a linear map $A \longrightarrow \mathrm{Hom}(B, C)$. Therefore we get a bijection of sets

$$\mathbf{Ab}(A \otimes B, C) \longleftrightarrow \mathbf{Ab}(A, \mathrm{Hom}(B, C))$$

The tensor product is unital, associative, and commutative. This means there is a unit object I and natural isomorphisms

$$l_A : I \otimes A \xrightarrow{\cong} A$$

$$r_A : A \otimes I \xrightarrow{\cong} A$$

$$a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

$$s_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

The unit object is the group of integers $\mathbb{Z}$. We can start with a tensor product of a bunch of groups and start applying these isomorphisms willy-nilly to regroup the parentheses and rearrange terms:

$$(A \otimes (B \otimes C)) \otimes D \xrightarrow{\cong} (A \otimes (C \otimes B)) \otimes D \xrightarrow{\cong} ((A \otimes C) \otimes B) \otimes D \xrightarrow{\cong} \ldots$$

If we ever come back to the expression we started with, then the composition of the maps we applied becomes the identity map. (This means that the isomorphisms l, r, a, s are *coherent*. Heuristically, this means we can drop the parentheses around the tensor products without getting into trouble.) If we carefully rewrite the above properties of $(\mathbf{Ab}, \mathbb{Z}, \otimes, \mathrm{Hom}, l, r, a, s)$ using only notation from category theory, we get the concept of a *closed symmetric monoidal category.* If we drop everything involving Hom, then $(\mathbf{Ab}, \mathbb{Z}, \otimes, l, r, a, s)$ gives a *symmetric monoidal category.*

We've been building up to a statement about **HoSpectra**, so here it is: **HoSpectra** is a closed symmetric monoidal category. Its unit object is the sphere spectrum $\mathbb{S}$. Its tensor product is called the smash product $\wedge$, since it is based on the smash product of based spaces

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

(In some papers it's called the *left derived smash product* $\wedge^L$, to distinguish it from the smash product $\wedge$ in **Spectra**.) Its internal hom is denoted $F(X, Y)$. So if X, Y, and Z are spectra, there are natural coherent isomorphisms in **HoSpectra**

$$\begin{aligned} \mathbb{S} \wedge X &\cong X \\ (X \wedge Y) \wedge Z &\cong X \wedge (Y \wedge Z) \\ X \wedge Y &\cong Y \wedge X \\ [X \wedge Y, Z] &\cong [X, F(Y, Z)] \\ F(\mathbb{S}, X) &\cong X \\ F(X \wedge Y, Z) &\cong F(X, F(Y, Z)) \end{aligned} \tag{1}$$

Exercises.

- Show that the last two isomorphisms follow from the first four. (Use the Yoneda Lemma.)
- Define natural maps

$$X \wedge F(X, Y) \longrightarrow Y$$
$$X \longrightarrow F(Y, X \wedge Y)$$

- Let's define suspension and looping more explicitly:

$$\Sigma X = (\Sigma^\infty S^1) \wedge X$$
$$\Omega X = F(\Sigma^\infty S^1, X)$$

  Using the above isomorphisms, together with the fact that Σ^∞ is a functor, prove that suspension and looping are adjoint:

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

  and construct the operation on either of these two sets that turns it into an abelian group. (Can we prove that Σ and Ω are inverses yet? Why not?)

- Prove that there are natural isomorphisms

$$(\Sigma X) \wedge Y \cong \Sigma(X \wedge Y) \cong X \wedge (\Sigma Y)$$
$$\Omega F(X, Y) \cong F(\Sigma X, Y) \cong F(X, \Omega Y)$$

There are many concepts in algebra that have an analogue in the world of spectra. Here's an important example. Start with a symmetric monoidal category **C** and an object M of **C**. If we can give a "multiplication" morphism $\mu : M \otimes M \longrightarrow M$ that is associative

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{\mathrm{id} \otimes \mu} & M \otimes M \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ M \otimes M & \xrightarrow{\mu} & M \end{array}$$

and a "unit" morphism $i : I \longrightarrow M$ of this multiplication

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{i \otimes \mathrm{id}} & M \otimes M & \xleftarrow{\mathrm{id} \otimes i} & M \otimes I \\ & {}_{l_M}\searrow & \downarrow \mu & \swarrow_{r_M} & \\ & & M & & \end{array}$$

then we say that (M, μ, i) is a *monoid* in **C**. Fun fact: a monoid in **Ab** is the same thing as a ring! By analogy, we call a monoid in **HoSpectra** a *ring spectrum.* If you understand this example, you should be able to define a *commutative monoid* in **C**. If you define it correctly, a commutative monoid in **Ab** will be a commutative ring.

Here's a list of some symmetric monoidal categories, and common names for monoids in those categories:

Category	Tensor Product	Unit	Monoid	Commutative Monoid
Set	$\times$	$\{*\}$	Monoid	Commutative Monoid
Ab	$\otimes$	$\mathbb{Z}$	Ring	Commutative Ring
Graded Ab$^{(1)}$	$\otimes$	$\mathbb{Z}$	Graded Ring	Commutative Graded Ring
Graded Ab$^{(2)}$	$\otimes$	$\mathbb{Z}$	Graded Ring	Skew-Commutative Ring*
$\mathbf{Mod}_R$	$\otimes_R$	R	R-algebra	Commutative R-algebra
Top	$\times$	$\{*\}$	Topological Monoid	Commutative Topological Monoid
$\mathbf{Top}_*$	$\wedge$	S^0	Based Topological Monoid	Based Commutative Topological Monoid
Spectra	$\wedge$	$\mathbb{S}$	(Strict) Ring Spectrum	(Strict) Commutative Ring Spectrum
HoSpectra	$\wedge^{(L)}$	$\mathbb{S}$	Ring Spectrum (up to homotopy)	Commutative Ring Spectrum (up to homotopy)

* There are two common conventions for the symmetry isomorphism for graded abelian groups:

$$\begin{aligned} a \otimes b &\mapsto b \otimes a \\ a \otimes b &\mapsto (-1)^{|a| \cdot |b|}(b \otimes a) \end{aligned}$$

Under the first convention, a commutative monoid is a commutative ring that happens to be graded. Under the second convention, a commutative monoid is a *skew-commutative ring.* This means that even-degree elements commute with everything, and odd-degree elements introduce a -1 when switched past each other. Sometimes skew-commutative rings are called graded-commutative or even just commutative, but don't confuse them with commutative graded rings like $\mathbb{Z}[x]$. Skew-commutative rings show up all over algebraic topology: the cohomology of a space $H^*(X)$ and the stable homotopy groups of spheres $\pi_*^S(S^0)$ are two examples. We're doing algebraic topology here, so we'll follow the second convention and work with skew-commutative rings.

Now we have a language that relates spectra to abelian groups. But we really want much more. Consider the diagram we gave at the beginning, with **Spectra** deleted because it doesn't always have a good smash product:

$$\begin{array}{ccccccc}
\mathbf{Top} & \xrightarrow{X\mapsto X_+} & \mathbf{Top}_* & & & & \mathbf{Ab} \\
\downarrow & & \downarrow & \searrow^{\Sigma^\infty} & & \swarrow^{G\mapsto HG} & \downarrow{\scriptstyle \text{0th degree}} \\
\mathbf{HoTop} & \longrightarrow & \mathbf{HoTop}_* & \xrightarrow{\Sigma^\infty} & \mathbf{HoSpectra} & \xrightarrow{\pi_*} & \mathbf{Graded\ Ab}
\end{array}$$

We claim that every functor in this diagram agrees with tensor products. To be more specific, if $F : C \longrightarrow D$ is any functor in the diagram, X and Y are objects of C, and I_C and I_D are the units of C and D, respectively, then there are natural transformations

$$F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$$
$$F(I_C) \longrightarrow I_D$$

that commute with the unit, associativity, and symmetry isomorphisms of C and D.

Exercise. Prove that a functor F with these properties takes monoids to monoids.

A functor F satisfying these properties is called *lax monoidal.* If the above maps are isomorphisms, then F is called *strong monoidal.* In the above diagram, every functor is at least lax monoidal. So if we start with a (commutative) monoid anywhere on the diagram, and follow any route, we end up at another (commutative) monoid. For example, if X is a ring spectrum, then $\pi_*(X)$ is a graded ring.

Exercises.

- Prove that the forgetful functor $\mathbf{Ab} \longrightarrow \mathbf{Set}$ is lax monoidal.
- Prove that its left adjoint, the "free abelian group on a set" construction, is strong monoidal.

Here's another example. Consider the one-point space $\{*\}$ in **Top**. This clearly forms a commutative monoid. Its image in **HoSpectra** is the sphere spectrum $\mathbb{S}$. Therefore $\mathbb{S}$ is a commutative ring spectrum! Applying π_*, we deduce that the stable homotopy groups of spheres $\pi_*^S(S^0) \cong \pi_*(\mathbb{S})$ form a skew-commutative ring.

To draw more conclusions about multiplication, we need to take a look at homology and cohomology.

1.4 Exact Sequences

One of the main goals in algebraic topology is to actually *calculate* the invariants we define for interesting objects. And one of the most basic tools for calculation is the exact sequence. It turns out that **HoSpectra** has a notion of "short exact sequences" of objects, which generalize the classical cofiber sequences

$$A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A$$

and the classical fiber sequences

$$\Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

To be more precise, we can form triples of objects (X, Y, Z), and triples of maps (f, g, h) of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

We call (X, Y, Z, f, g, h) a *triangle*. Now here is our claim. There is a collection of triangles in **HoSpectra**, called the *distinguished triangles*, that satisfy a few useful properties:

- Every cofiber sequence or fiber sequence in $\mathbf{Top}_*$ becomes a distinguished triangle in **HoSpectra**.
- For each distinguished triangle

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

  and each object W, there are long exact sequences of abelian groups

  $$\ldots \longrightarrow [W, X]_n \longrightarrow [W, Y]_n \longrightarrow [W, Z]_n \longrightarrow [W, X]_{n-1} \longrightarrow \ldots$$

  $$\ldots \longleftarrow [X, W]_n \longleftarrow [Y, W]_n \longleftarrow [Z, W]_n \longleftarrow [X, W]_{n-1} \longleftarrow \ldots$$

  Taking $W = \mathbb{S}$, we see that the stable homotopy groups form a long exact sequence

  $$\ldots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z) \longrightarrow \pi_{n-1}(X) \longrightarrow \ldots$$

- If (X, Y, Z, f, g, h) is distinguished and W is another object, then

  $$W \wedge X \xrightarrow{f} W \wedge Y \xrightarrow{g} W \wedge Z \xrightarrow{h} \Sigma(W \wedge X)$$
  $$F(W, X) \xrightarrow{f} F(W, Y) \xrightarrow{g} F(W, Z) \xrightarrow{h} \Sigma F(W, X)$$
  $$\Sigma^{-1} F(X, W) \xrightarrow{-h} F(Z, W) \xrightarrow{g} F(Y, W) \xrightarrow{f} F(X, W)$$

  are distinguished.

There are actually a lot more properties than the ones we mentioned. If we were to list them all, we would be able to say that **HoSpectra** forms a *triangulated category* [7]. Instead, we'll just take the above results and use them to study homology and cohomology.

1.5 Homology and Cohomology

The objects of **HoSpectra** define (reduced) homology and cohomology theories on $\mathbf{Top}_*$. To see this, take a based space X and a spectrum E in **HoSpectra**. Then the abelian groups

$$\widetilde{E}_n(X) = [\mathbb{S}, (\Sigma^\infty X) \wedge E]_n \cong \pi_n((\Sigma^\infty X) \wedge E)$$
$$\widetilde{E}^n(X) = [\Sigma^\infty X, E]_{-n} \cong \pi_{-n}(F(\Sigma^\infty X, E))$$

define an (extraordinary, reduced) homology theory and a cohomology theory.

Exercise. Use the statements from the last section to prove that these satisfy the axioms of a reduced (co)homology theory. Equivalently, show that

$$E_n(X) = \widetilde{E}_n(X_+)$$
$$E^n(X) = \widetilde{E}^n(X_+)$$

form an unreduced cohomology theory.

Now we can easily generalize from the homology of spaces to the homology of spectra. If Y and E are objects in **HoSpectra**, define the abelian groups

$$\widetilde{E}_n(Y) = [\mathbb{S}, Y \wedge E]_n \cong \pi_n(Y \wedge E)$$
$$\widetilde{E}^n(X) = [Y, E]_{-n} \cong \pi_{-n}(F(Y, E))$$

Exercises.

- Construct the above two isomorphisms, using only the properties we have already given for **HoSpectra**.
- Show that $E_*(\mathrm{pt}) \cong E^*(\mathrm{pt})$.
- If E is a ring spectrum, and X is a space, show that $E^*(X)$ is a graded ring. Define an unreduced cohomology theory E^* for spectra, and use it show that $E^*(Y)$ is a graded ring.
- If E is a commutative ring spectrum, show that $E^*(Y)$ is a graded skew-commutative algebra over the commutative ring $\widetilde{E}^*(\mathbb{S}) \cong E^*(\mathrm{pt})$.
- If E and Y are both ring spectra, show that $E_*(Y)$ is a graded ring. What happens when E is commutative?

So every object of **HoSpectra** gives an extraordinary cohomology theory, on CW-complexes or even on spectra. It turns out that the converse is true: every cohomology theory on CW-complexes extends to a cohomology theory on **HoSpectra**, and is represented by an object in **HoSpectra**. (This is *Brown Representability*. By the Yoneda Lemma, the representing object is unique up to isomorphism.)

Remark. There is sometimes ambiguity when we talk about (co)homology of spaces that are not homotopy equivalent to CW-complexes. Here we always assume that you take a CW approximation of such a space, and then you take the cellular cohomology of the result. This is represented in $\mathbf{HoTop}_*$ by the Eilenberg-Maclane space $K(G, n)$.

Let's consider $\widetilde{H}^*(X; G)$, the theory of ordinary (singular or cellular) cohomology with coefficients in an abelian group G. By the above statement, there is an object in **HoSpectra** called HG, the *Eilenberg-Maclane spectrum* associated to G, and a natural isomorphism

$$\widetilde{(HG)}^n(X) \cong \widetilde{H}^n(X; G)$$

The associated homology theories also agree: (why is this always true?)

$$\widetilde{(HG)}_n(X) \cong \widetilde{H}_n(X; G)$$

It turns out that $H : \mathbf{Ab} \longrightarrow \mathbf{HoSpectra}$ is (strong?) monoidal. So if R is a commutative ring, then HR is a commutative ring spectrum. The multiplication on $\widetilde{(HR)}^*(X)$ is just the cup product on $\widetilde{H}^*(X; R)$! If X is an unbased topological monoid, then the multiplication on $\widetilde{(HR)}_*(X_+)$ is the Pontryagin product on $\widetilde{H}_*(X_+; R) \cong H_*(X; R)$.

Similarly, there is a spectrum KU for complex K-theory, KO for real K-theory, MU for complex cobordism, and MO for real cobordism. There is a long list of interesting cohomology theories, and we won't try to list all of them. But we can still list infinitely many of them: any based space X becomes a cohomology theory $\Sigma^\infty X \in \mathbf{HoSpectra}$, whose groups are "shifted stable maps into X." Classically, we had to think of cohomology theories and the spaces we took cohomology of as different objects. In **HoSpectra**, we can think of them on equal terms, and state theorems that apply to both.

1.6 In Summary

The stable homotopy category **HoSpectra** has:

- A functor Σ^∞ coming in from based spaces.
- Suspension that is invertible up to natural isomorphism.
- Morphism sets $[X, Y]$ that are abelian groups, which extend to graded abelian groups $[X, Y]_*$. Composition of morphisms is graded and bilinear.
- Stable homotopy groups $\pi_*(X)$.
- A zero object $*$, and a natural isomorphism $X \vee Y \xrightarrow{\cong} X \times Y$ from coproducts to products.
- A unit object $\mathbb{S}$, a smash product $X \wedge Y$, and an internal hom $F(X, Y)$, together with some natural isomorphisms, that make it a closed symmetric monoidal category.
- Distinguished triangles that form long exact sequences of homotopy groups, and that agree with the smash product and internal hom.
- Objects which represent cohomology theories.

1.7 Atiyah Duality of Manifolds

Here's a geometric application of the above properties. Two based finite CW-complexes A, B are *strongly n-dual* if there is an embedding $\Sigma^k A \hookrightarrow S^{k+n+1}$ and a homotopy equivalence $\Sigma^l B \xrightarrow{\simeq} \Sigma^l(S^{k+n+1} - \Sigma^k A)$. By Alexander duality, this gives isomorphisms

$$\widetilde{H}_q(A) \cong \widetilde{H}^{n-q}(B)$$
$$\widetilde{H}^q(A) \cong \widetilde{H}_{n-q}(B)$$

If A and B are strongly n-dual, then there is a map

$$\Sigma^{k+l}(A \wedge B) \longrightarrow \Sigma^{k+l} S^n$$

such that if we pull back the top-dimensional cohomology class of S^n to $A \wedge B$, the slant product with this class gives isomorphisms as above

$$\widetilde{H}_q(A) \cong \widetilde{H}^{n-q}(B)$$
$$\widetilde{H}^q(A) \cong \widetilde{H}_{n-q}(B)$$

In this case we say that A and B are simply *n-dual.* These two (different) notions are often both called *Spanier-Whitehead duality.*

It turns out that Spanier-Whitehead duality is much easier to state and work with in the stable homotopy category. We say that two objects A, B of **HoSpectra** are *dual* if there is a map $A \wedge B \longrightarrow \mathbb{S}$ inducing isomorphisms

$$A \cong F(B, \mathbb{S})$$
$$B \cong F(A, \mathbb{S})$$

Notice the parallel with vector spaces, where $\mathrm{Hom}(V, k)$ is defined to be the dual of V, and the double dual of V is naturally isomorphic to V itself. Two spectra are *n-dual* if there is a map $A \wedge B \longrightarrow \Sigma^n \mathbb{S}$ inducing isomorphisms

$$A \cong F(B, \Sigma^n \mathbb{S}) \quad \Leftrightarrow \quad \Sigma^{-n} A \cong F(B, \mathbb{S})$$
$$B \cong F(A, \Sigma^n \mathbb{S}) \quad \Leftrightarrow \quad \Sigma^{-n} B \cong F(A, \mathbb{S})$$

If A and B are n-dual spaces, then $\Sigma^\infty A$ and $\Sigma^\infty B$ are n-dual spectra. So we can import the entire theory of Spanier-Whitehead duality to the stable homotopy category, and the statements become cleaner.

If M is an m-manifold, then there is a smooth embedding $e : M \hookrightarrow \mathbb{R}^n$ for sufficiently large n. It follows that M_+ and $\mathbb{R}^n - M$ are (strongly) $(n-1)$-dual. But then the Thom space of the normal bundle M^ν is homotopy equivalent to the suspension of $\mathbb{R}^n - M$, so M_+ and M^ν are n-dual. This was classically called “Atiyah duality”. The n-duality map can be described explicitly as

$$M^\nu \wedge M_+ \longrightarrow S^n$$
$$\mathbb{R}^n / (\mathbb{R}^n - \nu_\epsilon(M)) \wedge M_+ \longrightarrow \mathbb{R}^n / (\mathbb{R}^n - B_\epsilon(0))$$
$$(x, y) \mapsto x - e(y)$$

As an immediate corollary, we get isomorphisms in **HoSpectra**

$$\Sigma^\infty M^\nu \simeq F(M_+, \Sigma^n \mathbb{S}) \quad \Rightarrow \quad M^{-TM} := \Sigma^{-n} \Sigma^\infty M^\nu \simeq F(M_+, \mathbb{S})$$

coming from the Alexander map above. This is *Atiyah duality* in the stable homotopy category. From this, and the properties discussed in previous sections, we can take any cohomology theory E in **HoSpectra** and get isomorphisms

$$\widetilde{E}_q(M^\nu) \cong E^{n-q}(M)$$
$$\widetilde{E}^q(M^\nu) \cong E_{n-q}(M)$$

(Recall that the tilde means the theory is reduced, and $E_q(M) := \widetilde{E}_q(M_+)$.)

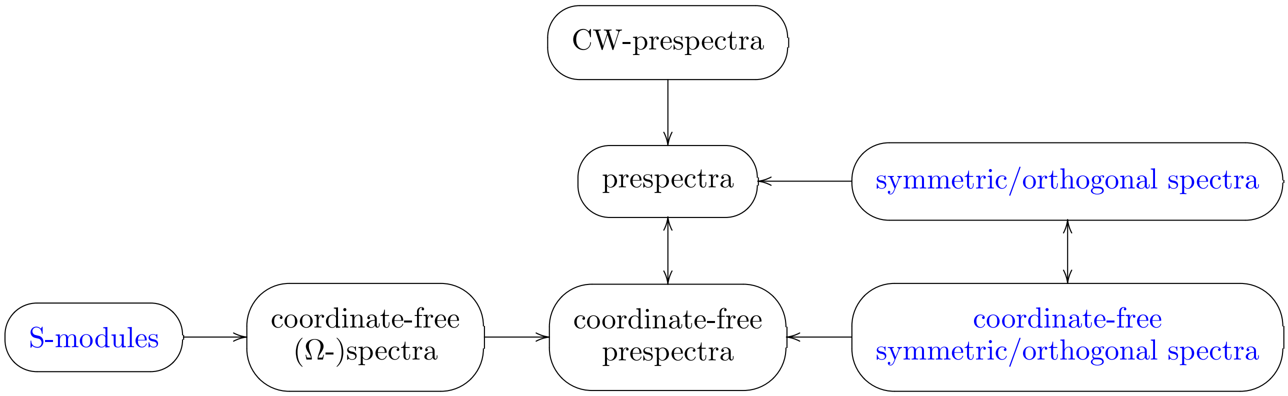
Notice that we have not assumed any kind of orientability for M. If M is orientable in the cohomology theory E, then applying the Thom isomorphism to these gives Poincaré duality in E:

$$E_{m-q}(M) \cong E^q(M)$$

We can put a product on M^{-TM} that gives the intersection product on homology; then the Atiyah duality isomorphism is an isomorphism of ring spectra [2]. As an easy consequence, Poincaré duality takes the intersection product on $E_{m-q}(M)$ to the cup product on $E^q(M)$.

2 HoSpectra: How Do We Construct It?

Now that we've made a wish list of all the properties we desire in the category **HoSpectra**, we'll give some actual constructions of this category. These constructions all require a category called **Spectra**, which could be the category of prespectra, CW-prespectra, symmetric spectra, orthogonal spectra, coordinate-free spectra of various kinds, or S-modules. We give them here with forgetful functors:



The blue-colored categories have good smash products in **Spectra** before descending to the stable homotopy category **HoSpectra**.

2.1 The Boardman Category

A *prespectrum* E is a sequence of based spaces $E_0, E_1, E_2, \ldots$ along with *structure maps* $\Sigma E_n \longrightarrow E_{n+1}$. We call E_n the nth *level* of E. A map of prespectra $f : X \longrightarrow Y$ is a sequence of maps $f_n : X_n \longrightarrow Y_n$ that commute with the structure maps:

$$\begin{array}{ccc} \Sigma X_n & \longrightarrow & X_{n+1} \\ \downarrow {\scriptstyle \Sigma f_n} & & \downarrow {\scriptstyle f_{n+1}} \\ \Sigma Y_n & \longrightarrow & Y_{n+1} \end{array}$$

A *CW-prespectrum* is a prespectrum E with the following properties: Each level E_n is a CW-complex. One of the 0-cells is chosen to be the basepoint. Therefore, the reduced suspension of each cell D^m is D^{m+1}, glued to the suspensions of the lower-dimensional cells. Therefore we can view ΣE_n as a CW-complex with one $(m+1)$-cell for every m-cell of E_n other than the basepoint. Using this cell structure on ΣE_n, we require that the structure map $\Sigma E_n \hookrightarrow E_{n+1}$ be the inclusion of a subcomplex.

Now every k-cell of E_n becomes a $(k+1)$-cell of E_{n+1}, a $(k+2)$-cell in E_{n+2}, etc. We call this a *stable* $(k-n)$*-cell*. It's clear that we can have stable m-cells for all integer values of m, so a CW-prespectrum is like a CW-complex in which we have somehow allowed negative-dimensional cells.

Our first definition of **HoSpectra** is a category **Board** whose objects are CW-prespectra, and whose morphisms are "eventually-defined maps up to eventually-defined homotopy". More precisely, each map $f : X \longrightarrow Y$ is a map on each stable m-cell of X that is defined on the $(m-n)$-cell in E_n for all sufficiently large values of n. Of course, the maps on different cells have to agree with the attaching maps of those cells. So if X has finitely many stable cells, then the map is eventually defined on all of X_n. In general, though, a map need not ever be defined on all of X_n for sufficiently large n.

Define a functor $\Sigma^\infty : \mathbf{HoTop}_* \longrightarrow \mathbf{Board}$ by taking each based CW-complex X to the prespectrum whose nth level is $\Sigma^n X$, and whose structure maps are the identity. (This functor extends to all of $\mathbf{HoTop}_*$ by some categorical nonsense, because everything there is isomorphic to a CW-complex.)

This construction is called the *Boardman category*. Historically, it was the first construction of the stable homotopy category. The description given here agrees with the one given by Adams in his classic notes [1].

Exercises.

- Describe the zero object $* := \Sigma^\infty(\{\text{pt}\})$ and the sphere spectrum $\mathbb{S} := \Sigma^\infty S^0$.
- Define the suspension functor $\Sigma : \mathbf{Board} \longrightarrow \mathbf{Board}$ so that it agrees with suspension in $\mathbf{HoTop}_*$.
- Define the shift functor $\text{sh} : \mathbf{Board} \longrightarrow \mathbf{Board}$ by $(\text{sh}E)_n = E_{n+1}$, with the obvious structure maps. Show that sh is a functor, and has an inverse up to natural isomorphism.
- Give a natural isomorphism between Σ and sh.
- Describe the abelian group structure of $[X, Y]$ and $[X, Y]_n$. In particular, this gives us the stable homotopy groups $\pi_n(X) := [\mathbb{S}, X]_n$.
- Prove that **Board** has all of the properties that we claimed for **HoSpectra** in section 1.2. Note that Σ is easy to work with, but Ω is quite tricky. This motivates the category that we define in the next section.
- Let **Board′** be defined as above, but we require that the attaching map of every cell in E_n to be a based map $S^{m-1} \longrightarrow X$, instead of an unbased map. (This is the definition given in a book by Switzer.) Give an equivalence of categories between **Board** and **Board′**.

2.2 All Prespectra

Let **Prespectra** denote the category of all prespectra (not just the CW ones) with maps that are defined on every level (not just eventually defined). So an object in **Prespectra** is a sequence of spaces $\{E_n\}_{n=1}^\infty$, together with maps $\Sigma E_n \longrightarrow E_{n+1}$. Notice that these maps always have adjoints

$E_n \longrightarrow \Omega E_{n+1}$. We say that E is a *(weak) Ω-spectrum* if these adjoints are all weak homotopy equivalences. We still define the stable homotopy groups of X to be $\pi_k(X) = \mathrm{colim}_k \pi_{n+k}(X_k)$. Notice that if X is a weak Ω-spectrum, then $\pi_k(X) = \pi_k(X_0)$.

Exercises.

- Let X be any prespectrum. Construct a CW-prespectrum $\widetilde{X}$ and a map of prespectra $\widetilde{X} \longrightarrow X$ that is a weak homotopy equivalence on each level.
- Construct a (weak) Ω-spectrum $\hat{X}$, and a map $X \longrightarrow \hat{X}$ that induces isomorphisms on the stable homotopy groups $\pi_*(X) \xrightarrow{\cong} \pi_*(\hat{X})$.

Now we can define **HoPrespectra** to have the same objects as **Prespectra**, but morphisms from X to Y are $[\widetilde{X}, \hat{Y}]$. Note that each map or homotopy is defined on *every* level, not just eventually defined. Intuitively, this will agree with the Boardman category because an eventually-defined map $\widetilde{X}_n \longrightarrow \hat{Y}_n$ can always be looped to give $\widetilde{X}_0 \longrightarrow \Omega^n \widetilde{X}_n \longrightarrow \Omega^n \hat{Y}_n \simeq \hat{Y}_0$. Now we have a good concrete description of the maps in the homotopy category, but unfortunately, in this description it is difficult to describe how we compose the maps. One way to circumvent this is to redefine the maps as zig-zags $X \longleftarrow A \longrightarrow B \longleftarrow \ldots \longrightarrow Y$, where the backwards maps are required to be π_*-isomorphisms. But this has problems of its own, so if we need to work with these objects on a very explicit level, we have to use the language of model categories.

There is a model structure on **Prespectra** with the following description, found in [5]. A cofibration is a retract of a relative stable cell complex. Here the stable cells are defined as in the previous section, but we allow ourselves to attach lower-dimensional cells to higher-dimensional ones. A weak equivalence is a map inducing isomorphisms on the stable homotopy groups. A fibration is a map $E \longrightarrow B$ such that every level $E_n \longrightarrow B_n$ is a Serre fibration, and in the square

$$\begin{array}{ccc} E_n & \longrightarrow & \Omega E_{n+1} \\ \downarrow & & \downarrow \\ B_n & \longrightarrow & \Omega B_{n+1} \end{array}$$

the natural map $E_n \longrightarrow B_n \times_{\Omega B_{n+1}} \Omega E_{n+1}$ is a weak homotopy equivalence. Now we can pass to **HoPrespectra**, and this category is isomorphic to the one defined above (although technically we didn't actually say above how to compose the maps).

Does **HoPrespectra** have any advantages over **Board**? In some contexts, it's simpler to work with. For example, to define the loopspace ΩX of a prespectrum X, we want to take $(\Omega X)_n = \Omega(X_n)$. In **Prespectra**, this is easy. In **Board**, we have to take functorial CW approximations, making the construction more obscure.

More importantly, we want to get a definition of smash product. However, in either **Board** or **HoPrespectra**, we define smash products by taking an arbitrary sequence $p(n) \longrightarrow \infty$ such that $(n - p(n)) \longrightarrow \infty$, and setting $(X \wedge Y)_n = X_{p(n)} \wedge Y_{n-p(n)}$. If we fix one such sequence, this defines a smash product that has all of the properties that we claimed in section 1. Unfortunately, this relies on a non-canonical choice of sequence, and we hate choices because they make things hard in practice. Fortunately, there exist some tricks for making it canonical. Here is one such trick:

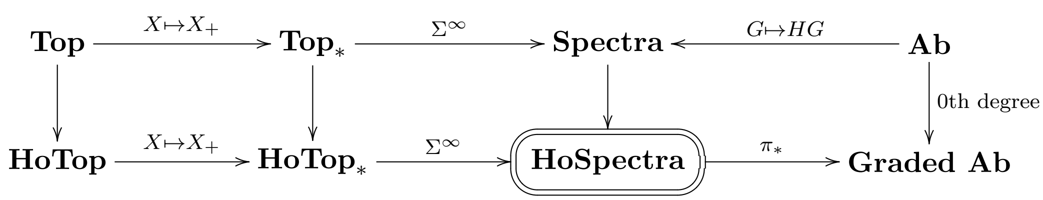
2.3 Symmetric and Orthogonal Spectra

A *symmetric spectrum* E is a sequence of based spaces $E_0, E_1, E_2, \ldots$ such that E_n has a Σ_n-action and such that the composite

$$S^p \wedge E_q \longrightarrow S^{p-1} \wedge E_{1+q} \longrightarrow \cdots \longrightarrow S^1 \wedge E_{(p-1)+q} \longrightarrow E_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant. A map $f : X \longrightarrow Y$ between symmetric spectra is a sequence of maps $f_n : X_n \longrightarrow Y_n$ that agree with suspension, such that f_n is Σ_n-equivariant. This defines a category called $\mathbf{Sp}^\Sigma$. To define an *orthogonal spectrum*, we take the above definition and replace Σ_n with $O(n)$ everywhere; this gives a category $\mathbf{Sp}^O$. (The $O(n)$ actions must be continuous.) We have an inclusion of groups $\Sigma_n \hookrightarrow O(n)$, so an orthogonal spectrum defines a symmetric spectrum.

It turns out that, unlike prespectra, symmetric (or orthogonal) spectra form a closed symmetric monoidal category. If we let **Spectra** denote either symmetric or orthogonal spectra, then we have the diagram



and every functor is at least lax monoidal. So we can define monoids in **Spectra**, which then become monoids in **HoSpectra**. These two notions are not the same. A monoid in **Spectra** is a *symmetric/orthogonal ring spectrum*, whereas a monoid in **HoSpectra** is just a ring spectrum "up to homotopy." We've neglected to actually describe **HoSpectra** in either of these two cases, and we'll continue to neglect this while we discuss the closed symmetric monoidal structure.

Let's describe this structure more explicitly. The unit object is the sphere spectrum $\mathbb{S}_n = S^n$. The mapping space is

$$F(X,Y)_n \subset \prod_i F(X_i, Y_{i+n})$$

the subspace of all collections of Σ_i-equivariant maps $\{X_i \longrightarrow Y_{i+n}\}_i$ that commute with suspension. Notice that $F(X,Y)_0$ is just the based space of all maps of symmetric spectra. The smash product is

$$(X \wedge Y)_n = \bigvee_{p+q=n} \Sigma_{p+q_+} \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)/\sim$$

The quotient relation identifies the images of the two maps

$$\begin{array}{ccccc} \Sigma_{(p+q+r)_+} \wedge_{\Sigma_q \times \Sigma_{p+r}} (X_q \wedge Y_{p+r}) & \longleftarrow & \Sigma_{(p+q+r)_+} \wedge (S^p \wedge X_q \wedge Y_r) & \longrightarrow & \Sigma_{(p+q+r)_+} \wedge_{\Sigma_{p+q} \times \Sigma_r} (X_{p+q} \wedge Y_r) \\ (\sigma \circ \tau_{q,p}, x, sy) & \longleftarrow & (\sigma, s, x, y) & \longrightarrow & (\sigma, sx, y) \end{array}$$

Here $(s, x) \mapsto sx$ is shorthand for the structure map

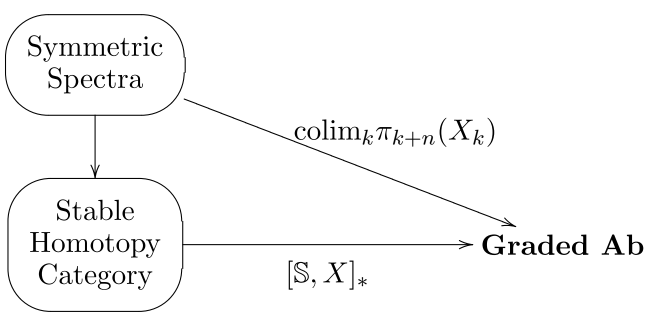
$$S^p \wedge X_q \cong \Sigma^p X_q \longrightarrow X_{p+q}$$

and $\tau_{q,p}$ is a permutation in Σ_{p+q+r} moves the first block of q elements past the second block of p elements and leaves the last block of r elements alone.

Let's describe this more heuristically. S^p has p sphere coordinates, X_q has q sphere coordinates, and Y_r has r sphere coordinates. They are naturally arranged with the p coordinates first, then the q coordinates, then the r coordinates. The permutation σ takes this natural arrangement and gives us the arrangement we desire. Now if we smash S^p into X_q, we get a space X_{p+q} with $(p+q)$ sphere coordinates, still lined in order with the p coordinates first and the q coordinates second. So in $X_{p+q} \wedge Y_r$, the p coordinates come first, then the q coordinates, then the r coordinates. Applying σ, we again get the desired arrangement of sphere coordinates.

However, if we smash S^p into Y_r, we get $X_q \wedge Y_{p+r}$. The q coordinates come first, then the p coordinates, then the r coordinates. Applying σ, we get the wrong arrangement. We fix the problem by applying $\sigma \circ \tau_{q,p}$ instead. The $\tau_{q,p}$ pulls the p coordinates back to the beginning where they belong. Therefore $\sigma \circ \tau_{q,p}$ gives us the correct arrangement of sphere coordinates. We remember to include $\tau_{q,p}$ by feeling a pang of guilt whenever we try to move S^p past X_q. The permutation $\tau_{q,p}$ alleviates that guilt.

To recap, symmetric (or orthogonal) spectra form a closed symmetric monoidal category. We can define a symmetric ring spectrum to be a monoid object in this category; this always descends to a monoid object in the homotopy category. Unfortunately, symmetric spectra sometimes have the "wrong" homotopy groups. If we try to define π_n of a symmetric spectrum X as $\text{colim}_{k\to\infty}\pi_{k+n}(X_k)$, then we get a diagram



This diagram does NOT commute, so the naïve homotopy groups $\pi_n(X) = \text{colim}_{k\to\infty}\pi_{k+n}(X_k)$ are not equal to our original definition $\pi_n(X) = [\mathbb{S}, X]_n$. Moreover, the naïve homotopy groups do not define a monoidal functor into **Graded Ab**. Therefore, the "correct" definition of homotopy groups is $\pi_*(X) = [\mathbb{S}, X]_*$. Fortunately, our two definitions coincide for the class of "semistable" symmetric spectra, as defined in [8].

Every orthogonal spectrum gives a "semistable" symmetric spectrum, so the naïve definition of π_* gives the right answer when X is an orthogonal spectrum. So orthogonal spectra enjoy the convenience of having a good smash product and internal hom, and the naïve definition of their homotopy groups is the correct one. The smash product of orthogonal spectra is defined as above, but with $O(n)$ everywhere instead of Σ_n.

We sometimes find it necessary to calculate the homotopy groups of a smash product of orthogonal spectra $X \wedge Y$. Using the above definition directly, this looks like a nightmare. Fortunately, (using [5]) these homotopy groups are equal to the ones we obtain from the "handicrafted smash product" of X and Y as prespectra. We can actually calculate these homotopy groups as the colimit of the following commuting grid of abelian groups:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \text{colim} = \pi_k(X \wedge Y) \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle -} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle -} & & \\
\pi_{2+k}(X_0 \wedge Y_2) & \xrightarrow{+} & \pi_{3+k}(X_1 \wedge Y_2) & \xrightarrow{+} & \pi_{4+k}(X_2 \wedge Y_2) & \xrightarrow{+} & \pi_{5+k}(X_3 \wedge Y_2) & \xrightarrow{+} & \cdots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle -} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle -} & & \\
\pi_{1+k}(X_0 \wedge Y_1) & \xrightarrow{+} & \pi_{2+k}(X_1 \wedge Y_1) & \xrightarrow{+} & \pi_{3+k}(X_2 \wedge Y_1) & \xrightarrow{+} & \pi_{4+k}(X_3 \wedge Y_1) & \xrightarrow{+} & \cdots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle -} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle -} & & \\
\pi_k(X_0 \wedge Y_0) & \xrightarrow{+} & \pi_{1+k}(X_1 \wedge Y_0) & \xrightarrow{+} & \pi_{2+k}(X_2 \wedge Y_0) & \xrightarrow{+} & \pi_{3+k}(X_3 \wedge Y_0) & \xrightarrow{+} & \cdots
\end{array}
$$

Here $(+)$ means that we use the usual suspension homomorphism, and $(-)$ means that we negate it. The signs are explained by the fact that a new sphere coordinate must be switched past X_p before it can be smashed into Y_q. Note that the colimit can be computed in at least three different ways: we can compute the colimit of each column and then take the colimit of the results, or we could do the same thing with rows, or we could take a path from the bottom-left corner out to infinity that eventually reaches each row and column, and take the colimit along that path.

How do we go to **HoSpectra**? We need a notion of an unbased cell; we get it by creating a "free" symmetric/orthogonal spectrum out of the map of based spaces $S^{n-1}_+ \hookrightarrow D^n_+$. (Here S^{n-1}_+ is a sphere with a disjoint basepoint, NOT the upper hemisphere of S^{n-1}.) Then there's a model structure in which the cofibrations are the retracts of the relative cell complexes, the fibrations are the levelwise fibrations $E \longrightarrow B$ giving homotopy pullbacks

$$
\begin{array}{ccc}
E_n & \longrightarrow & \Omega E_{n+1} \\
\downarrow & & \downarrow \\
B_n & \longrightarrow & \Omega B_{n+1}
\end{array}
$$

and the weak equivalences $X \longrightarrow Y$ are the maps that induce isomorphisms $[Y, E] \xrightarrow{\cong} [X, E]$ for every (weak) Ω-spectrum E. (These will coincide with the idea that $X \longrightarrow Y$ gives isomorphisms on the "correct" homotopy groups, but to define the "correct" groups we need to actually construct **HoSpectra** first.) This is the model structure given in [5]; another one is given in [8].

It's important to keep in mind that the smash product $\wedge^L$ on **HoSpectra** is not the same as $\wedge$ on **Spectra**; it is "left derived." There is a natural transformation $X \wedge^L Y \longrightarrow X \wedge Y$, which is an isomorphism when one of the factors is "flat." The unit object is still the sphere spectrum $\mathbb{S}$. This is discussed more in [8].

2.4 Coordinate-Free Spectra

At the risk of introducing additional confusion, we can talk about symmetric and orthogonal spectra in a "coordinate-free" way. We'll do this and also discuss other notions of coordinate-free spectrum that appear in the literature.

If A is a finite set, let $\mathbb{R}^A$ denote the space of all functions $A \longrightarrow \mathbb{R}$, and S^A its the one-point compactification. A *coordinate-free symmetric spectrum* is an assignment of a space $X(A)$ to each finite set A (in some appropriate universe), and a map $S^{B-i(A)} \wedge X(A) \xrightarrow{\xi_i} X(B)$ to each inclusion $i : A \hookrightarrow B$. The identity map $A \hookrightarrow A$ must induce the identity $S^0 \wedge X(A) \longrightarrow X(A)$, and for each composition $A \xrightarrow{i} B \xrightarrow{j} C$ the evident diagram commutes:

$$\begin{array}{ccc} S^{C-j(B)} \wedge S^{B-i(A)} \wedge X(A) & \xrightarrow{\xi_i} & S^{C-j(B)} \wedge X(B) \\ \downarrow & & \downarrow \xi_j \\ S^{C-j(i(A))} \wedge X(A) & \xrightarrow{\xi_{j\circ i}} & X(C) \end{array}$$

Exercise. Let $\mathbf{n}$ be the finite set $\{1, \ldots, n\}$. If X is a coordinate-free symmetric spectrum, construct an ordinary symmetric spectrum whose levels are $\{X(\mathbf{n})\}_{n=0}^{\infty}$.

Since every finite set is isomorphic to some $\mathbf{n}$, it's also possible to go backwards and turn any symmetric spectrum into a coordinate-free one. So the theory of coordinate-free symmetric spectra is essentially the same as the the theory of symmetric spectra.

If V is an inner product space, let S^V denote its one-point compactification. A *coordinate-free orthogonal spectrum* is an assignment of a space $X(V)$ to each finite-dimensional inner product space V, and a map $S^{W-i(V)} \wedge X(V) \xrightarrow{\xi_i} X(W)$ to each linear isometric inclusion $i : V \hookrightarrow W$. Here $W - i(V)$ is the orthogonal complement of $i(V) \subset W$. The maps must depend continuously on i. To state this precisely, let $O(V, W)$ be the space of linear isometries $V \hookrightarrow W$, and let $O(V, W)^{W-V}$ be the Thom space of the canonical bundle over the Grassmannian $O(V, W)$, whose fiber over i is $W - i(V)$. Then we require that the following map be continuous:

$$O(V, W)^{W-V} \wedge X(V) \longrightarrow X(W)$$

The identity map $V \hookrightarrow V$ must induce the identity $S^0 \wedge X(V) \longrightarrow X(V)$, and for each composition $V \longrightarrow V' \longrightarrow V''$ the evident diagram commutes. As above, if X is a coordinate-free orthogonal spectrum, then the sequence of spaces $X_{\mathbb{R}^n}$ forms an orthogonal spectrum that captures all the information in X up to isomorphism. We can also avoid set-theoretic difficulties by declaring that our "universe" is just some infinite-dimensional real inner product space $U \cong \mathbb{R}^\infty$, and that we only consider the finite-dimensional subspaces $V \subset U$. Note however that we work with all linear isometric injective maps $V \hookrightarrow W$, not just the inclusions of subspaces $V \subset W \subset U$.

Next we briefly describe coordinate-free (Ω-)spectra, a theory that should not be confused with the theory of symmetric/orthogonal spectra. As above, we fix a universe $U \cong \mathbb{R}^\infty$ with an inner product, and for each finite-dimensional $V \subset U$ we let S^V be its one-point compactification. If K is any based space, let $\Omega^V K = F(S^V, K)$ be space of based maps in the (CGWH) compact-open topology. A *coordinate-free prespectrum* X associates to every finite-dimensional subspace $V \subset U$ a based space $X(V)$, and to every inclusion $V \subset W$ of subspaces a continuous map $S^{W-V} \wedge X(V) \longrightarrow X(W)$. Equivalently, there is a continuous map $X(V) \longrightarrow \Omega^{W-V} X(W)$. We have identity and composition axioms: the inclusion $V \subset V$ must induce the identity map $X(V) \longrightarrow X(V)$, and a triple of inclusions $V \subset V' \subset V''$ yield three maps that must agree. A *coordinate-free spectrum* X is a prespectrum for which the maps $X(V) \longrightarrow \Omega^{W-V} X(W)$ are homeomorphisms. These are discussed in classic notes by Lewis, May and Steinberger [4].

Remark. Note that coordinate-free (Ω-)spectra only have maps for *inclusions* of spaces $V \subset W$, whereas coordinate-free orthogonal spectra have maps for *every injective map* $V \hookrightarrow W$ that preserves the inner product. It is not difficult to see that the only spectrum satisfying both definitions is the zero object, $X(V) = *$ for all V. (To do this, consider the $O(V)$-equivariant map

$$X(V) \xrightarrow{\cong} \Omega^{W-V} X(W)$$

when $\dim(W - V) = 2$.)

Unfortunately, even coordinate-free spectra do not themselves form a closed symmetric monoidal category. But we can pass to a subcategory of "S-modules" that does. This is done in work of Elmendorf, Kriz, Mandell, and May [3]. The homotopy category of S-modules is then equivalent to **Board**, **HoPrespectra**, $\mathbf{HoSp}^\Sigma$, and $\mathbf{HoSp}^O$ as defined in previous sections. S-modules in the sense of EKMM are not as elementary as orthogonal spectra, though they have at least two advantages: all the objects are already fibrant, so the smash product $\wedge^L$ on the homotopy category is simpler than it is for symmetric/orthogonal spectra. The second advantage is that an E_∞ operad is built into the definition of the smash product, so that a commutative monoid in **HoSpectra** is a strict E_∞ ring spectrum in some appropriate sense. (In symmetric spectra, a commutative monoid in the homotopy category is weaker than an E_∞ ring spectrum, and there is an obstruction theory to get from one to the other; see [8] for details.)

In the long run, we need to be able to use coordinate-free constructions (S-modules or orthogonal spectra) because they naturally generalize to the G-equivariant setting.

2.5 A_∞ and E_∞ Rings

An A_∞ symmetric ring spectrum becomes a ring spectrum in the homotopy category, but not conversely. In the category of S-modules, a ring spectum in the appropriate homotopy category is exactly the same thing as an A_∞ ring.

Talk about FSPs and coordinate-free rings.

3 The Parametrized World

Loosely, a *parametrized spectrum* or *fibered spectrum* is an object over some (CGWH) topological space B such that the "fiber" over each point $b \in B$ is a spectrum, using one of the definitions we gave in the previous section. If B is a point, a fibered spectrum over B should just be a spectrum. These objects are very useful tools for collecting together unstable information (in the base B) with stable information (in the fibers). As a basic application, one can prove a version of twisted Poincaré duality that is much more powerful and general than the usual one using ordinary (co)homology with twisted coefficients.

There are at least three approaches to parametrized spectra: the May-Sigurdsson approach uses coordinate-free orthogonal spectra, but there is another approach using S-modules, and other approaches using ∞-categories and/or homotopy sheaves.

3.1 Notes on the May-Sigurdsson Approach

The reference for this entire section is [6]. Let B be an unbased (CGWH) topological space. An ex-space over B is a topological space X (which for technical reasons must be a k-space but need not be weak Hausdorff) together with maps $B \longrightarrow X \longrightarrow B$ that compose to the identity. The category of such spaces is denoted $\mathcal{K}_B$. This category has products $X \times_B Y$, quotients $X/_B Y$, wedge sums $X \vee_B Y$, and smash products $X \wedge_B Y$, and each of these constructions does the obvious thing on each fiber. It also has mapping spaces $F_B(X,Y)$, which on each fiber is the mapping space of fibers $F(X_b, Y_b)$, but its construction is a bit subtle.

Let's define the "right" model structure on these guys, the qf-model structre. This structure is "compactly generated" in the sense that there is a collection of cells I and trivial cells J that are compact in some sense, such that the cofibrations are the retracts of relative I-cell complexes, the acyclic fibrations have the RLP with respect to maps in I, the acyclic cofibrations are the retracts of the relative J-cell complexes, and the fibrations have the RLP with respect to maps in J. It's also "well grounded," in the sense that there is a forgetful functor to spaces (total space), and the model structure on spaces interacts in the correct way with the qf-model structure on $\mathcal{K}_B$. As a consequence, arguments like the Puppe cofibration sequence go through. (Other model structures run into problems with this.)

To define the qf-model structure, we first define the f maps. On p.80, we see that f-cofibrations,

fibrations, and weak equivalences are fiberwise HEP, HLP, and homotopy equivalence. There are $\bar{f}$-cofibrations, but these end up being f-cofibrations that are also closed inclusions. From p.84, these give an f-model structure on spaces over B and ex-spaces over B.

Now we can give the qf-model structure on ex-spaces over B. From p.101, an f-disc is a disc $D^n \longrightarrow B$ such that $S^{n-1} \hookrightarrow D^n$ is an f-cofibration; this is morally the same as saying that the map $D^n \longrightarrow B$ is constant on some collar neighborhood of the boundary of D^n. A relative f-disc is a diagram of f-cofibrations over B

$$\text{upper hemisphere} \longrightarrow S^n \longrightarrow D^{n+1}$$

Equivalently, D^{n+1} and its lower hemisphere are both f-discs. Then I is the collection of f-discs $S^{n-1} \longrightarrow D^n$ (with a disjoint section attached) and J is the collection of relative f-discs upper hemisphere $\longrightarrow D^{n+1}$ (with a disjoint section attached). These collections generate the cofibrations and acyclic cofibrations of the qf-model structure. The qf-equivalences are the weak homotopy equivalences on total spaces. This is enough to determine the qf model structure: the cofibrations are the retracts of the f-disc complexes, whereas the qf-fibrations have the usual lifting property with respect to every relative f-disc. Every qf-cofibrant object is f-cofibrant, $\bar{f}$-cofibrant, and q-cofibrant. Every f-fibrant object is qf-fibrant. Every qf-fibrant object is a quasifibration, i.e. for every point $b \in B$ there is a long exact sequence of homotopy groups. So for the fibrant objects, the homotopy groups of each fiber capture the homotopy type.

Now we'll move from spaces $\mathcal{K}_B$ to spectra $\mathcal{S}_B$. Fix a universe U. For each finite-dimensional subspace V, let S^V_B be the fiberwise one-point compactification of the trivial bundle $B \times V \longrightarrow B$. Now a *parametrized coordinate-free orthogonal spectrum* is an assignment of an ex-space $X(V)$ to each finite-dimensional inner product space V, and a map of ex-spaces $S_B^{W-i(V)} \wedge_B X(V) \xrightarrow{\xi_i} X(W)$ to each linear isometric inclusion $i : V \hookrightarrow W$. As before, the maps must depend continuously on i:

$$(O(V,W)^{W-V} \times B) \wedge_B X(V) \longrightarrow X(W)$$

As before, these maps also respect the identity and composition.

To construct the homotopy category $\mathbf{Ho}\mathcal{S}_B$, we restrict attention to orthogonal spectra whose levels $X(V)$ are well-grounded ($\bar{f}$-cofibrant and CGWH). Construct shift desuspensions $F_V : \mathcal{K}_B \longrightarrow \mathcal{S}_B$ just as in the nonparametrized case:

$$F_V(A)(W) = (O(V,W)^{W-V} \times B) \wedge_B A$$

Then the level model structure has as its weak equivalences the levelwise weak homotopy equivalences of total spaces over B. The cofibrations and acyclic cofibrations generated by the shift desuspensions of the f-discs and the relative f-discs, respectively. The stable model structure has weak equivalences the maps that induce isomorphisms on the stable homotopy groups of each fiber. The cofibrations are the same as in the level case; this is enough to determine the fibrations. The fibrant objects are levelwise qf-fibrant and are Ω-spectra in the sense that the maps from one space to fiberwise loops of the next is a weak homotopy equivalence on the total space.

Notice that if $B = *$ then we get the category of coordinate-free orthogonal spectra from a previous section, with the same stable model structure, yielding the same homotopy category.

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